Stability in matching with couples having non-responsive preferences*†

Shashwat Khare[1] and Souvik Roy[‡2]

[1]Quantitative Economics, Maastricht University
[2]Economics Research Unit, Indian Statistical Institute

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Abstract

We study many-to-one matching problems between institutions and individuals where an institution can possibly be matched to more than one individual. The matching market contains some couples, who view pairs of jobs as complements. Institutions' preferences over sets of individuals are assumed to satisfy responsiveness. However, couples' preferences over pairs of institutions are allowed to violate responsiveness. In this setting, first we assume that institutions have a common preference over individuals, and we provide

(i) a complete characterization of all preferences of couples such that a stable matching exists under the additional assumption that

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‡Corresponding author: souvik.2004@gmail.com

couples violate responsiveness in order to be matched at the same institution, and

(ii) a necessary and sufficient condition on the common preference of institutions so that a stable matching exists when couples can violate responsiveness in an arbitrary manner.

Next, we weaken the common preference assumption on institutions' preferences by requiring common preference only over the members of each couple. In this setting, we provide

(i) a complete characterization of all preferences of couples such that stable matching exists, and

(ii) a sufficient condition on the preferences of individuals' such that a stable matching exists.

KEYWORDS. many-to-one two-sided matching, stability, responsiveness, togetherness
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1 Introduction

In many different contexts, there is a centralized matching procedure by which individuals on one side of the market are matched to institutions on the other side of the market. These include the market for lawyers in Canada, children to schools in the USA, doctors and senior-level health-care professionals in several countries, etc. A matching is pairwise stable if there does not exist any institution-individual pair that can block it by getting matched together, such that both of them are better off compared to their original matching.

Roth[9] showed that it is possible to design mechanisms which incentivize only one side of the market to truthfully reveal their preferences. However, the results on stability have been more promising. The received doctrine is that stable matchings do exist under appropriate domain restrictions. In particular, institutions have to view individuals as substitutes and individuals must care only about institutions to which they are matched.

Klaus and Klijn[6] showed that a stable matching exists at every preference profile when couples' preferences satisfy responsiveness. Responsiveness means that a couple is better off when any member of the couple is matched to a more preferred institution keeping the other member fixed.[1] However, Kojima, Pathak and Roth[7] pointed out that responsiveness is not satisfied in their data set because couples show strong preference to be matched to institutions situated in the same geographical area.

It was first pointed out by Roth[10] that the existence of a stable matching is *not* guaranteed at *every* preference profile in the presence of couples in the labour market. This can happen because couples may view pairs of jobs as complements, which would violate responsiveness. This motivates us to provide a characterization of preference profiles at which a stable matching exists despite the violation of responsiveness.

We consider a specialized matching problem between a set of hospitals and a set of doctors including some couples. We focus on the issue of existence of stable matchings with couples. We first look at the scenario when all hospitals have a common preference over individual doctors. This can be justified if hospitals rank doctors according to their grades of some common examination. Hospitals' preferences over sets of doctors are derived from the common preference by using responsiveness. Responsiveness says that for two allocations of a hospital, which differ by exactly one doctor, the hospital prefers the allocation with the better doctor. Note that there can be several responsive preferences over sets of doctors for a given preference over individual doctors. In our model, different hospitals are allowed to have different responsive preferences (over sets of doctors).

[1] Roth and Sotomayer[11] introduced the substitutability condition and showed that it is sufficient to ensure the existence of a stable matching. Hatfield and Milgrom[5] showed that the substitutes condition, which is a natural extension of substitutability, is also sufficient for stability. Later, Hatfield and Kojima[4] showed that the bilateral substitutes condition is sufficient for stability and that responsiveness implies bilateral substitutes.

Each individual doctor has a strict preference over hospitals. The preference of a couple is derived from the preferences of the members of the couple. We let couples' preferences violate responsiveness in an appropriate way to capture their willingness to be matched together. Thus, in spirit of Dutta and Massó[1], we assume that a couple prefers to be matched at the same hospital rather than being matched to different hospitals. We show that if hospitals have a common preference over doctors, then a stable matching exists if and only if each couple's preference does not violate responsiveness with respect to the more preferred (according to the common preference of hospitals) member of the couple.

Next, we consider the scenario where couples are allowed to have arbitrary preferences over pairs of hospitals. This captures situations where two particular doctors prefer to stay away from each other. We show that a stable matching exists in this scenario if and only if for each couple, one of the following happens: (i) either the members of the couple are ranked consecutively or (ii) there is at most one doctor ranked in-between the members and one member of the couple is ranked at the bottom of the common preference of hospitals.

Finally, we weaken the assumption that hospitals have a common preference over individual doctors by requiring that they have a common preference only over the members of each couple. We provide a necessary and sufficient condition on couples' preferences for the existence of a stable matching. Next, we look at this problem from hospitals' point of view and provide a sufficient condition on the preferences of hospitals such that a stable matching exists for any couples' preferences where the couples can violate responsiveness to be together.

The rest of the paper is organized as follows. We formally introduce the model in Section 2. In Section 3, we investigate the existence of a stable matching when couples' preferences can violate responsiveness in order to be together and hospitals have a common preference over individual doctors. In Section 4, we consider the situation where couples' preferences are unrestricted (and hospitals have a common preference). Finally, in Section 5, we relax the assumption of a common preference of hospitals and provide sufficient conditions for the existence of a stable matching.

2 The framework

We consider many-to-one matching between doctors and hospitals. We denote by H the set of hospitals. We use the notation $\bar{H}$ to denote $H \cup \{\emptyset\}$. The interpretation of $\emptyset$ is that if some doctor is matched to $\emptyset$, then that doctor is practically unmatched (that is, it is not assigned to any hospital). Each hospital $h \in H$ has a finite capacity, denoted by $\kappa_h \geq 2$.

We denote by D the set of doctors. We assume that $D = F \cup M \cup S$, where F, M, S are pairwise disjoint sets of doctors. We denote the doctors in F by $\{f_1, \ldots, f_k\}$ and those in M by $\{m_1, \ldots, m_k\}$, for some $k \in \mathbb{N}$.[2] This, in particular, means that F and M have the same number of doctors. The doctors in F and M together form fixed couples, whereas the doctors in S are not part of any couple. We call the doctors in S single doctors and those in M or F non-single doctors. We denote the set of couples by $C = \{\{f_1, m_1\}, \ldots, \{f_k, m_k\}\}$ and a generic couple by $c = \{f, m\}$.

Throughout this paper, we assume $|H| \geq 2$, $|D| \geq 4$ and $|C| \geq 1$. That is, there are at least two hospitals and four doctors including at least one couple. We also assume that the total number of vacancies in all hospitals in H is equal to the total number of doctors available, that is, $\sum_{h \in H} \kappa_h = |D|$.
[3]

An allocation of a couple $c = \{f, m\}$ is an element (h, h') of $\bar{H}^2$ where hospitals h and h' are matched with doctors f and m, respectively. As we have already mentioned, here one or both of h and h' might be empty, which

[2]By $\mathbb{N}$, we denote the set of natural numbers $\{1, 2, \ldots\}$.

[3]The situation where there are more (or less) doctors than the total capacity of hospitals can be dealt with by considering dummy hospitals (or, doctors) and by making suitable modification of the preferences over these dummies.

would mean that the corresponding doctor(s) is(are) not matched with any hospital.

For notational convenience, we do not use braces for singleton sets.

2.1 Matching

A matching is an allocation of the doctors over the hospitals such that the total allocation of doctors in a hospital does not exceed its capacity and a doctor is allocated to at most one hospital (that is, is allocated to exactly one hospital or no hospital). Below, we provide a formal definition of this.

DEFINITION **1** *A matching on $H \cup D$ is a mapping μ on $H \cup D$ such that:*

(i) for all $h \in H$, $\mu(h) \subseteq D$ with $|\mu(h)| \leq \kappa_h$,

(ii) for all $d \in D$, $\mu(d) \in \bar{H}$,

(iii) for all $d \in D$ and all $h \in H$, $\mu(d) = h$ if and only if $d \in \mu(h)$.

2.2 Preferences

In this subsection, we introduce the notion of preferences of hospitals and doctors. We also propose certain restrictions on those.

For a set X, we denote by $\mathbb{L}(X)$ the set of linear orders, that is, complete, reflexive, transitive, and antisymmetric binary relations over X. An element $R \in \mathbb{L}(X)$ is called a preference over X and P is the strict part of R. Since a preference is antisymmetric, xRy implies either $x = y$ or xPy. For $P \in \mathbb{L}(X)$ and $k \leq |X|$, we define $r_k(P)$ as the k-th ranked alternative in P, that is, $r_k(P) = x$ if and only if $|\{y \in X : yRx\}| = k$. Moreover, for $P \in \mathbb{L}(X)$ and $x \in X$, we define by $r(x, P)$ the rank of x in P, that is, $r(x, P) = k$ if and only if $r_k(P) = x$.

2.2.1 Preferences of hospitals

For any hospital $h \in H$, a preference of h over individual doctors, denoted by $\tilde{P}_h$, is defined as an element of $\mathbb{L}(D \cup \{\emptyset\})$.

We assume $d\tilde{P}_h\emptyset$ for all $d \in D$ and all $h \in H$. That is, a hospital always prefers to have a doctor than having a vacant position.

For a hospital h, the feasible sets of doctors (given its capacity) is defined as $\{D' \subseteq D : |D'| \leq \kappa_h\}$. A preference over feasible sets of doctors of a hospital h is an element of $\mathbb{L}(\{D' \subseteq D : |D'| \leq \kappa_h\})$. In what follows we discuss how a preference of a hospital over individual doctors is extended to that over feasible sets of doctors. We introduce the notion of responsiveness in this context.

Responsiveness captures the idea of separability that is used in the context of extending preferences over individual dimensions to that over over multi-dimensions. Roughly speaking, responsiveness says that hospitals always prefer it when they get a better doctor (or set of doctors). For example, consider a preference $\tilde{P}_h$ of a hospital over the individual doctors $\{d_1, d_2, d_3, d_4\}$, where $d_1\tilde{P}_h d_2 \tilde{P}_h d_3 \tilde{P}_h d_4$. Then, responsiveness says that the pair (d_1, d_2) will be preferred to the pair (d_1, d_3), the set (d_1, d_2, d_4) will be preferred to the set (d_1, d_3, d_4), etc. in the extension of $\tilde{P}_h$ over feasible sets of doctors. It is important to note that responsiveness does not say how the hospital h will compare certain sets of doctors, for instance, the pairs (d_1, d_4) and (d_2, d_3). So, one can have a responsive extension of $\tilde{P}_h$ where the first pair is preferred to the second, and another where the second pair is preferred to the first. Below, we provide a formal definition of responsive extension.

Definition 2 *Let h be a hospital with capacity κ_h and let $\tilde{P}_h$ be a preference of h over individual doctors. Then, a preference P_h of h over feasible sets of doctors satisfies responsiveness with respect to $\tilde{P}_h$ if*

(i) the restriction of P_h over individual doctors coincides with $\tilde{P}_h$, that is, for all $d, d' \in D \cup \{\emptyset\}$, $dP_h d'$ if and only if $d\tilde{P}_h d'$, and

(ii) for all $D' \subsetneq D$ and all $D_1, D_2 \subseteq D \setminus D'$ such that $|D' \cup D_1| \leq \kappa_h$ and $|D' \cup D_2| \leq \kappa_h$, we have $(D' \cup D_1) P_h (D' \cup D_2)$ if and only if $D_1 P_h D_2$.

Next, we define the notion of common preference of hospitals over individual doctors. As the name suggests, this simply says that all hospitals have the same preference over the individual doctors. Such a preference can be viewed as the common ranking of the individual doctors based on the grades of some common examination, etc. Note that hospitals may, in principle, differ on the extension of this common preference over sets of feasible doctors.

DEFINITION **3** *Let $\{P_h\}_{h \in H}$ be a collection of preferences of hospitals over feasible sets of doctors and let $P^0_{hp} \in \mathbb{L}(D \cup \{\emptyset\})$. Then, $\{P_h\}_{h \in H}$ is said to satisfy Common Preference over Individual doctors (CPI) with respect to P^0_{hp} if for all $h \in H$, P_h is responsive with respect to P^0_{hp}.*

Unless mentioned otherwise, we assume CPI for every collection of preferences of hospitals. Whenever we consider a collection of preferences satisfying CPI with respect to P^0_{hp}, we assume for ease of presentation that the indexation of the doctors in couples is such that $f P^0_{hp} m$ for every couple $c = \{f, m\} \in C$, and that of the couples in $C = \{\{f_1, m_1\}, \ldots, \{f_k, m_k\}\}$ is such that $m_1 P^0_{hp} m_2 P^0_{hp} \ldots P^0_{hp} m_k$. This is without of loss of generality as we consider only one CPI at every given context.

It is worth mentioning that although our aforementioned assumption is without loss, the restrictions we put on the female member of a couple in the later part of the paper are basically imposed on the "commonly preferred" member of a couple, and thus, do not have anything to do with any particular member (say, female) of a couple.

2.2.2 Preferences of doctors

Every doctor has a preference over hospitals including the 'empty' hospital $\emptyset$. Thus, a preference P_d of a doctor $d \in D$ is an element of $\mathbb{L}(\bar{H})$. We assume $hP_d\emptyset$ for all $h \in H$ and all $d \in D$. In other words, all doctors prefer being matched to some hospital than being unemployed. Now, we proceed to define the preference of a couple based on the preferences of the members in it.

Preferences of couples

Each couple has a preference over the pairs of hospitals. Thus, a preference P_c of a couple c is an element of $\mathbb{L}(\bar{H}^2)$. Recall that an allocation (h_1, h_2) for a couple $c = \{f, m\}$ means that f is matched with h_1 and m is matched with h_2. For a couple $c = \{f, m\}$ with preference P_c, and a hospital $h \in \bar{H}$, the conditional preference of m given h_1, $P_{m|h_1}$, is defined as the following preference of m: for all $h_1, h_2 \in \bar{H}$, $h_1 P_{m|h} h_2$ if and only if $(h, h_1)P_c(h, h_2)$.

As we have discussed earlier, in this paper we intend to deviate from responsiveness in a 'minimal' way and study its consequences on stability. We assume that a preference of a couple is responsive except in the situations where both the members of the couple get to stay together at some hospital. For instance, if f prefers h_1 to h_2 and m prefers h_2 to h_1, then, in contrast to responsiveness where the pair (h_1, h_2) should have been preferred to both the pairs (h_1, h_1) and (h_2, h_2), we allow for the couple $\{f, m\}$ to prefer (h_1, h_1) or (h_2, h_2) or both to the pair (h_1, h_2). Clearly, we allow this because at the allocation (h_1, h_1) or (h_2, h_2), the members of the couple can benefit from staying together. We call this 'preference for togetherness'. Note that we still assume that a couple prefers an allocation where both its members are matched to another where at least one member is unmatched.

To define the notion of responsiveness violated for togetherness, we use the notion of responsiveness for couples' preferences. This notion of responsiveness is exactly the same as that for a preference of a hospital. However, for the sake of completeness, we present the formal definition of this here.

Definition **4** *Let $c = \{f, m\}$ be any couple and suppose P_f and P_m are the preferences of f and m, respectively. Then, a preference $P_c \in \mathbb{L}(\bar{H}^2)$ of the couple c is called responsive with respect to P_f and P_m if, for all $h, h_1, h_2 \in \bar{H}$, we have* *(i)* $(h, h_1)P_c(h, h_2)$ *if and only if* $h_1 P_m h_2$*, and*

(ii) $(h_1, h)P_c(h_2, h)$ *if and only if* $h_1 P_f h_2$*.*

Now, we are ready to define the notion of responsiveness violated for togetherness for preferences of couples.

DEFINITION **5** *Let* $c = \{f, m\}$ *be any couple and let* P_f *and* P_m *be the preferences of* f *and* m*, respectively. Then, a preference* $\bar{P}_c \in \mathbb{L}(\bar{H}^2)$ *of* c *satisfies* responsiveness violated for togetherness (RVT) *if there exists a responsive (with respect to* P_f *and* P_m*) preference* P_c *of* c *such that*

(i) for all $h \in H$ *and all* $(h_1, h_2) \in \bar{H}^2$*, if* $(h, h)P_c(h_1, h_2)$ *then we have* $(h, h)\bar{P}_c(h_1, h_2)$*, and*

(ii) for all $(h, h'), (h_1, h_2) \in \bar{H}^2$ *such that* $h \neq h'$ *and* $h_1 \neq h_2$*, we have* $(h, h')P_c(h_1, h_2)$ *if and only if* $(h, h')\bar{P}_c(h_1, h_2)$*.*

Note that RVT implies that couples' preferences can violate responsiveness only in order to be together at some hospital. Also, by taking $h_1 = h_2$ in Condition (i) of Definition 5, it follows that for all $h, h' \in H$, $(h, h)P_c(h', h')$ if and only if $(h, h)\bar{P}_c(h', h')$.

2.2.3 Preference profiles and matching problems

A preference profile is a collection of preferences for all the doctors in D, all the couples in C, and all the hospitals in H, where hospitals' preferences are assumed to be responsive. Thus, a preference profile, denoted by $\underset{\sim}{P}$, is a collection of preferences $(\{\underset{\sim}{P}_d\}_{d \in D}, \{\underset{\sim}{P}_c\}_{c \in C}, \{\underset{\sim}{P}_h\}_{h \in H})$ where for all $d \in D$, $c \in C$ and $h \in H$, $\underset{\sim}{P}_d$ is a preference of doctor d, $\underset{\sim}{P}_c$ is a preference of couple c and $\underset{\sim}{P}_h$ is a responsive preference over feasible sets of doctors of hospital h, respectively.

By a matching problem, we mean a set of hospitals with corresponding capacities, a set of doctors with its partition into the sets F, M, and S, and a preference profile.

Throughout the paper, we maintain the following notational terminology. Whenever we refer to a given collection of preferences of hospitals or couples in any context, we use the superscript 0. For instance, we have used the notation P^0_{hp} to denote a CPI, and later we will use P^0_H and P^0_C to denote a given collection of preferences of hospitals and couples, respectively.

2.3 Stability

Our model is formally equivalent to a many-to-many matching market as a couple looks for two positions and hospitals have at least two positions. Thus, one can have different notions of stability based on different types of permissible blocking coalitions.[4]

Blocking pairs can be a hospital and a single doctor or a pair of hospitals and a couple.

We say a hospital h is 'interested' in a set of doctors D' at a matching μ if there is $D'' \subseteq \mu(h)$ such that $\{(\mu(h) \setminus D'') \cup D'\} P_h \mu(h)$. In other words, a hospital is interested in a set of doctors at a matching if it prefers to appoint those doctors by possibly removing some of its existing/matched doctors (to adjust its capacity). Similarly, we say a doctor d (couple c) is interested in a hospital h (pair of hospitals (h, h')) at a matching μ if $hP_d\mu(d)$ $((h, h')P_c\mu(c))$. Note that if a hospital is interested in a set of doctors or a doctor (couple) is interested in a hospital (pair of hospitals) at a matching μ, then it must be that they are not (already) matched at μ.

Now, we define the notion of (individual) blocking between a hospital and a single doctor.

Definition **6** *For a single doctor s, a hospital h, and a matching μ, we say (h, s) blocks μ if both h and s are interested in each other at μ.*

Thus, a hospital and a single doctor block a matching if they are not matched together at that matching but prefer to be so.

[4]See Roth[9], Roth[10], Konishi and Ünver[8], Echenique and Oviedo[2] for some alternative notions of stability in many to many matching problems.

Next, we define the notion of blocking between a pair of hospitals and a couple. A pair of hospitals and a couple, who are not already matched, block a matching if the couple prefers to be matched with that pair of hospitals, and the hospitals from that pair who are getting a new doctor from the couple are interested in getting it. Thus, the crucial thing here is that one of the members of the blocking couple might already be matched with one of the hospitals in the blocking pair. In that case, the other hospital must be interested in getting the other member of the couple. One might think that this case can be captured by our notion of (individual) blocking between the 'other hospital' and the 'other doctor'. Firstly, note that we have such notion of blocking only between hospitals and single doctors. Secondly, even if we define the notion of blocking between arbitrary (not necessarily single) individual doctor and hospital, that would not capture this situation as the other doctor might not be interested in the other hospital according to his/her individual preference but can be interested according to his/her couple preference.

DEFINITION **7** *For a couple $c = \{f, m\}$, a pair of hospitals (h_f, h_m), and a matching μ, we say $((h_f, h_m), c)$ blocks μ if c is interested in (h_f, h_m) at μ, and*

(i) if $h_x \neq h_y$ and $\mu(x) \neq h_x$ for all $x \in \{f, m\}$, then h_f is interested in f and h_m is interested in m,

(ii) if $h_x \neq h_y$ and $h_x = \mu(x)$ and $h_y \neq \mu(y)$ for $x, y \in \{f, m\}$, then h_y is interested in y,

(iii) if $h_f = h_m = h$, then h is interested in $\{f, m\}$.

It is worth mentioning that the blocking notion takes complementarity of a couple being accepted into account (by allowing the notion of a hospital being interested in a couple) but it does not take the couple into account when accepting single doctors and possibly removing members of a couple. In other words, there is an asymmetry here.

We consider this asymmetry in our model since it is not practical for big institutions like hospitals to consider the possibility of losing a member of a couple while removing the other member. This is because this possibility depends on factors like which hospital the removed member will join, whether the couple prefers to be together in that hospital, etc. Clearly, such situations can only be modeled by using a farsighted notion of blocking, which would complicate the model considerably.

DEFINITION **8** *A matching μ is stable if it can not be blocked.*

REMARK **1** *By our assumption that each hospital finds each doctor acceptable and each doctor finds each hospital acceptable, every stable matching is* individually rational.

REMARK **2** *To ease the presentation, for some couple $c = \{f, m\} \in C$ and hospitals $h_f, h_m \in H$, whenever a matching μ is blocked by $((h_f, h_m), c)$ where one of the members $x \in \{f, m\}$ of the couple was already in the corresponding hospital h_x (that is, $h_x = \mu(x)$), we simply say that μ is blocked by the other pair (h_y, y); $y \neq x \in \{f, m\}$.*

2.4 Two well-known algorithms

In this section, we present a well-known algorithm called doctor proposing deferred acceptance algorithm (DPDA).[5] However, for our purpose, we modify this algorithm slightly. We use this modified algorithm to match hospitals with doctors. In what follows, we give a short description of DPDA, where each doctor d has a preference P_d over hospitals and each hospital h has a preference P_h over feasible sets of doctors.

DPDA: This algorithm has multiple stages. In stage 1, each doctor $d \in D$ proposes to his/her most preferred hospital according P_d. Each hospital $h \in H$ provisionally accepts the most preferred collection of doctors according

[5]This directly follows from the well known algorithm given by Gale and Shapley[3].

to P_h. If a hospital h receives more than κ_h proposals, then it keeps its most preferred κ_h many doctors from these proposals and rejects all others. Having defined stages $1, \ldots, k$, the stage $k+1$ is defined in the following way: Each unmatched (till stage k) doctor d proposes to his/her most preferred hospital from the remaining set of hospitals who have not rejected him/her in any of the earlier stages. If a hospital, whose provisional list of accepted doctors is less than its capacity, receives one or more fresh proposal, then it continues to add to its accepted list (till its capacity). However, if a hospital h, whose provisional list of doctors is equal to its capacity, receives one or more fresh proposal from more preferred doctors, then it accepts these fresh proposals by rejecting same number of relatively worse (according to P_h) doctors that it provisionally accepted earlier. The algorithm finally terminates when each doctor is either matched with some hospital or has been rejected by all hospitals.

REMARK **3** *In DPDA, each individual doctor proposes according to his/her individual preference. Therefore, couples do not play any role in it.*

Now, we present another well-known algorithm called serial dictatorship algorithm (SDA). We give a short description of SDA where hospitals' preferences satisfy CPI with respect to P^0_{hp}. Recall that unless otherwise mentioned, we assume that hospitals' preferences satisfy CPI property. That is, they have a common ranking, denoted by P^0_{hp}, over individual doctors.

SDA: In SDA, the highest-ranked doctor according to P^0_{hp} chooses his/her most-preferred hospital, and in general, the j-th ranked doctor according to P^0_{hp} chooses his/her most preferred hospital among the hospitals with available vacancy after all the better (with rank less than j) doctors have made their choices.

Our next remark is a standard result in matching theory.

REMARK **4** *Both DPDA and SDA produce the same matching when hospitals' preferences satisfy CPI.*

3 Stability is not guaranteed under RVT

In this section, we explore the possibility of having a stable matching when couples' preferences satisfy RVT property, that is, they are allowed to violate responsiveness in order to be together. First, we show by the means of an example that, even when hospitals' preferences satisfy CPI, a stable matching is not guaranteed in such situations.

Example **1** Suppose there are two hospitals each having capacity 2, two single doctors, and two other non-single doctors forming a couple. Formally, suppose $H = \{h_1, h_2\}$, $\kappa_{h_1} = \kappa_{h_2} = 2$, and $D = \{s_1, s_2, f, m\}$ where $c = \{f, m\}$ is the only couple. Consider the preference profile given in Table 1. Here, both hospitals have a common preference over individual doctors which is denoted by P^0_{hp}.

We do not present the preferences of hospitals over feasible sets of doctors because that does not play any role in this example.

P^0_{hp}	P_{s_1}	P_{s_2}	P_f	P_m	P_c
f	h_2	h_1	h_2	h_1	(h_1, h_1)
s_1	h_1	h_2	h_1	h_2	(h_2, h_1)
s_2					(h_2, h_2)
m					(h_1, h_2)

Table 1

Note that the couple's preference violates responsiveness in order to be together at h_1 since the pair (h_1, h_1) is preferred to the pair (h_2, h_1) in its preference.

Now we show that there does not exist a stable matching at the given preference profile. Suppose on the contrary that μ is a stable matching at this profile. Since the couple prefers to be matched to any pair of hospitals than having a member unmatched, it follows that both the members of the couple must be matched with some hospitals at the matching μ. We consider all such allocations of the couple c, and show that blocking happens for each of these allocations.

(i) Suppose $\mu(c) = (h_1, h_1)$.
Since $h_1 P_{s_2} h_2$ and $s_2 P^0_{hp} m$, (h_1, s_2) blocks μ.

(ii) Suppose $\mu(c) = (h_2, h_1)$.
Because $h_2 P_{s_1} h_1$ and $s_1 P^0_{hp} s_2 P^0_{hp} m$, it must be that $\mu(s_1) = h_2$. Moreover, by responsiveness, $\{f, m\} P_{h_1} \{s_2, m\}$. This, together with the fact that $(h_1, h_1) P_c (h_2, h_1)$, implies $((h_1, h_1), c)$ blocks μ.

(iii) Suppose $\mu(c) = (h_2, h_2)$.
Since $h_2 P_{s_1} h_1$ and $s_1 P^0_{hp} m$, (h_2, s_1) blocks μ.

(iv) Suppose $\mu(c) = (h_1, h_2)$.
Because $h_2 P_{s_1} h_1$ and $s_1 P^0_{hp} s_2 P^0_{hp} m$, we have $\mu(s_1) = h_2$. Moreover, by responsiveness $\{f, m\} P_{h_2} \{s_1, m\}$. This, together with the fact that $(h_2, h_2) P_c (h_1, h_2)$, implies $((h_2, h_2), c)$ blocks μ.

Since Cases (i)-(iv) are exhaustive, it follows that there does not exist a stable matching at the preference profile given in Table 1.

4 Existence of stable matchings when couples' preferences satisfy RVT

In view of the fact that existence of stable matchings is not guaranteed when couples are allowed to violate responsiveness for togetherness, we search for additional conditions on couples' preferences so that the said existence is guaranteed.

Let $P^0_C = (\{P^0_d\}_{d \in D \setminus S}, \{P^0_c\}_{c \in C})$ be a given collection of preferences of non-single doctors (that is, the doctors in $D \setminus S$) and couples such that for all $c \in C$, P^0_c satisfies RVT. Given such a collection of preferences P^0_C, an extension of P^0_C refers to any preference profile where (i) the preferences of non-single doctors and couples are as given in P_C^0, and (ii) hospitals' preferences satisfy CPI with respect to some preference P_{hp}^0 over the individual doctors.

Recall that whenever hospitals' preferences over feasible sets of doctors satisfy CPI with respect to some preference P_{hp}^0 over individual doctors, we assume that $f_i P_{hp}^0 m_i$ for each couple $\{f_i, m_i\}$.

In what follows, we present a condition, called responsive for F (RF) property, that we use in describing situations where stable matchings exist. The RF property implies that couples' preferences are always responsive with respect to f. More precisely, if a couple moves together to a hospital from a pair of hospitals, then it must be that the f-member of the couple prefers that hospital to the hospital that he/she was originally matched with. In other words, compromise is always made by m in order for a couple $\{f, m\}$ to be together at some hospital.

It is worth emphasizing that the RF property is nothing particular about the female members of couples. This property basically means that the responsiveness can only be violated disfavoring the lesser preferred member of a couple. Since responsiveness is satisfied for the more preferred member of a couple, who we refer to as the female member, we use the term responsiveness for F.

Definition **9** *A collection of preferences P_C^0 is said to satisfy the Responsive for F (RF) property if for all $c = \{f, m\} \in C$ and all $h, h' \in H$, $(h, h) P_c^0 (h', h)$ implies $h P_f^0 h'$.*

Now, we present our first theorem which provides a necessary and sufficient condition for the existence of stable matchings at every possible extension of a given collection of preferences P_C^0. In particular, it says that a stable matching exists at every possible extension of P_C^0 if and only if P_C^0 satisfies the RF property. In the interest of readability, we present the if part and the only if part of this theorem separately.

Theorem 1 *(i) If P_C^0 satisfies the RF property, then a stable matching exists at every extension of P_C^0.*
(ii) If P_C^0 does not satisfy the RF property, then there always exists an extension of P_C^0 at which there is no stable matching.

The proof of this theorem is relegated to Appendix.

5 Existence of stable matchings when couples' preferences are unrestricted

In Section 4, we have considered the case where couples violate responsiveness for being matched together at some hospital and have provided a necessary and sufficient condition for the existence of stable matchings. In this section, we go beyond RVT and consider arbitrary violation of responsiveness of couples' preferences. In other words, we assume that a couple can have any preference over pairs of hospitals irrespective of the individual preferences of its members. Note that in our model a couple need not be a wife-husband pair, it only represents a pair of doctors who have a joint preference. For instance, we might have two jealous/competitive people who prefer to stay apart, and therefore have a joint preference. This justifies our consideration of arbitrary couples' preferences.

The existence of a stable matching cannot be anymore guaranteed in this setting in general. However, as we show, it can be assured by strengthening the CPI property of hospitals' preferences.

In what follows, we introduce the notion of strong CPI and show that it is both necessary and sufficient for the existence of a stable matching in this setting. First, we provide a verbal description of this property. Suppose that hospitals' preferences satisfy CPI with respect to P_{hp}^0. Roughly speaking, strong CPI ensures that the members of each couple are ranked 'very close' to each other. More precisely, it says that (i) if the m-member of a couple is not the worst doctor of D according to P_{hp}^0 and if there are enough doctors in D to fill (or exceed) the capacity of at least one hospital, then, in fact, the members of that couple must be ranked consecutively in P^0_{hp}, (ii) otherwise, there can be at most one doctor ranked in-between the members of the couple. Below, we provide the formal definition of this.

Recall that whenever hospitals' preferences are assumed to satisfy CPI with respect to some P^0_{hp}, we assume $fP^0_{hp}m$ for any couple $\{f,m\}$. Also, recall that we write $r(d,P^0_{hp})=k$ to mean that d has rank k in P^0_{hp}, that is, $r_k(P^0_{hp})=d$.

DEFINITION **10** *Let a collection of hospitals' preferences satisfy CPI with respect to P^0_{hp}. Then, the collection of preferences is said to satisfy strong CPI (SCPI) if for any couple $c=\{f,m\}\in C$,*

(i) $r(m,P^0_{hp})\neq|D|$ implies either $|\{d\in D: fP^0_{hp}dP^0_{hp}m\}|=0$ or $|\{d\in D: dP^0_{hp}m\}|<\kappa_h$ for all $h\in H$, and

(ii) $r(m,P^0_{hp})=|D|$ implies $|\{d\in D: fP^0_{hp}dP^0_{hp}m\}|\leq 1$.

A preference P_c of a couple $c\in C$ is unrestricted if it is an arbitrary element of $\mathbb{L}(\bar{H}^2)$ satisfying the only requirement that the couple prefers both its members to be matched to some hospital rather than having at least one member unmatched.

Suppose that hospitals' preferences satisfy CPI with respect to some preference P^0_{hp} over the individual doctors. We introduce the notion of extension of this preference P^0_{hp} to preference profiles. As the name suggests, an extension of P^0_{hp} to a preference profile is basically a preference profile where hospitals satisfy CPI with respect to P^0_{hp}. By an RVT extension of P^0_{hp}, we refer to any preference profile where couples' preferences satisfy RVT, and by an unrestricted extension of P^0_{hp}, we refer any preference profile where the couples' preferences are unrestricted. Of course, in both these extensions hospitals' preferences satisfy CPI with respect to P^0_{hp}.

Our next theorem provides a sufficient condition for the existence of a stable matching at every unrestricted extension of hospitals' preferences satisfying CPI.

THEOREM 2 *Let hospitals' preferences satisfy CPI with respect to P^0_{hp}. If P^0_{hp} satisfies SCPI, then a stable matching exists at every unrestricted extension of P^0_{hp}.*

The proof of this theorem is relegated to Appendix 7.

Now, we look at the converse of Theorem 2. It states the following: If hospitals' preferences satisfy CPI but violates SCPI, then there always exists an unrestricted extension of P^0_{hp} at which there is no stable matching. However, we prove a stronger version of this converse, where we show that when hospitals' preferences satisfy CPI but violates SCPI, one can find even an RVT extension of P^0_{hp} where there is no stable matching. Thus, under the said assumption, one does not have to look for an unrestricted extension to get hold of a profile with no stable matching.

THEOREM 3 *Let hospitals' preferences satisfy CPI with respect to P^0_{hp}. If P^0_{hp} does not satisfy SCPI, then there always exists an RVT extension of P^0_{hp} at which there is no stable matching.*

The proof of this theorem is relegated to Appendix 7.

The following corollary is immediate from Theorem 2 and (the stronger version of) Theorem 3.

COROLLARY 1 *Let hospitals' preferences satisfy CPI with respect to P^0_{hp}. Then a stable matching is guaranteed at every unrestricted extension of P^0_{hp} if and only if P^0_{hp} satisfies SCPI.*

6 Matching market with non-identical hospital preferences

In both Section 4 and Section 5, we have assumed that hospitals have identical preferences over the doctors. In this section, we relax this assumption and investigate the existence of stable matchings.

It is evident from Example 1 that a stable matching cannot be guaranteed in this setting unless we impose some additional conditions. One natural candidate for this is the RF property of couples' preferences. However, as we show in Example 2, that this property itself is not enough to ensure the existence of a stable matching at every profile.

Recall that a collection of preferences P^0_C satisfies the RF property if for all $c = \{f, m\} \in C$ and all $h, h' \in H$, $(h, h)P^0_c(h', h)$ implies $hP^0_f h'$.

EXAMPLE **2** Consider the matching problem where $H = \{h_1, h_2, h_3\}$ with $\kappa_{h_1} = \kappa_{h_2} = \kappa_{h_3} = 2$, $D = \{f, m, s_1, s_2, s_3, s_4\}$, and there is exactly one couple $c = \{f, m\}$ in C. The preferences of hospitals over individual doctors and those of individual doctors and couple are given in Table 1.

P_{h_1}	P_{h_2}	P_{h_3}	P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}	P_f	P_m	P_c
s_3	s_4	s_3	h_2	h_3	h_1	h_2	h_1	h_2	(h_1, h_2)
s_4	s_3	s_4	h_1	h_1	h_2	h_1	h_3	h_1	(h_1, h_1)
s_1	f	m	h_3	h_2	h_3	h_3	h_2	h_3	(h_1, h_3)
f	m	f							(h_3, h_3)
m	s_1	s_1							(h_3, h_2)
s_2	s_2	s_2							(h_3, h_1)
									(h_2, h_2)
									(h_2, h_1)
									(h_2, h_3)

Table 2

We show that there is no stable matching at this preference profile. Assume for contradiction that μ is a stable matching at that profile. Since μ is stable, $r_1(P_{s_3}) = h_1$ and $r_1(P_{h_1}) = s_3$ imply $\mu(s_3) = h_1$. Using similar logic, stability of μ implies $\mu(s_4) = h_2$. Moreover, because $s_1 P_h s_2$ for all $h \in H$, we must have $\mu(s_1) R_{s_1} \mu(s_2)$. Now, we consider all possible cases of couples' matching satisfying the above criteria and show that μ is blocked in each of those cases.

- Suppose $\mu(c) = (h_1, h_2)$. Since $h_1 P_{s_1} h_3$ and $s_1 P_{h_1} f$, μ is blocked by (h_1, s_1).

- Suppose $\mu(c) = (h_1, h_3)$. Since $(h_1, h_2) P_c (h_1, h_3)$ and $m P_{h_2} s_1$, μ is blocked by $((h_1, h_2), c)$.

- Suppose $\mu(c) = (h_3, h_3)$. Since $(h_1, h_2) P_c (h_3, h_3)$, $m P_{h_2} s_1$, and $f P_{h_1} s_2$, μ is blocked by $((h_1, h_2), c)$.

- Suppose $\mu(c) = (h_3, h)$ for some $h \in \{h_1, h_2\}$. Since $(h_3, h_3) P_c (h_3, h)$ and $m P_{h_3} s_2$, μ is blocked by $((h_3, h_3), c)$.

- Suppose $\mu(c) = (h_2, h_1)$. Since $h_1 P_{s_1} h_3$ and $s_1 P_{h_1} m$, μ is blocked by (h_1, s_1).

- Suppose $\mu(c) = (h_2, h_3)$. Since $(h_3, h_3) P_c (h_2, h_3)$ and $f P_{h_3} s_2$, μ is blocked by $((h_3, h_3), c)$.

Therefore, there is no stable matching at the preference profile given in Table 2.

In view of Example 2 we consider another property, namely the common preference over couple members property, to ensure the existence of a stable matching. As the name suggests, this property says that all hospitals have the same relative ranking over the members of each couple. Following our nomenclature, we assume that the common "better member" of each couple is the F-member and present this property by requiring that every hospital prefers the F-member of a couple to the M-member. Below, we provide a formal definition.

Definition **11** *A collection of preferences $P_H = \{P_h\}_{h \in H}$ of hospitals is said to satisfy the common preference over couple members (CPC) property if $f P_h m$ for all $c = \{f, m\} \in C$ and all $h \in H$.*

Now we show in Example 3, that even the RF property and the CPC property together are not sufficient to ensure stable matchings.

Example 3 In this example, we show that if a preference profile satisfies CPC, then there exists a preference extension with no stable matching even when the couples' preferences satisfy the RF property. Consider the matching problem where the set of hospitals, their capacities, and the set of doctors are as given in Example 2. The preferences of hospitals over individual doctors and those of individual doctors and the couple are given in Table 3. Note that hospitals' preferences satisfy CPC and the couples preferences satisfy responsiveness for F property.

P_{h_1}	P_{h_2}	P_{h_3}	P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}	P_f	P_m	P_c
s_3	s_4	s_3	h_2	h_3	h_1	h_2	h_1	h_2	(h_1, h_2)
s_4	s_3	s_4	h_1	h_1	h_2	h_1	h_3	h_1	(h_1, h_1)
s_1	f	f	h_3	h_2	h_3	h_3	h_2	h_3	(h_1, h_3)
f	m	m							(h_3, h_3)
m	s_1	s_1							(h_3, h_2)
s_2	s_2	s_2							(h_3, h_1)
									(h_2, h_2)
									(h_2, h_1)
									(h_2, h_3)

Table 3

The proof of the fact that there is no stable matching at the preference profile in Table 2 is similar to that in Example 2, and hence is omitted.

It follows from Example 3 that a stable matching does not exist at every preference profile even when we impose both the RF property for couples' preferences and the CPC property for hospitals' preferences, and further restrict that no doctor is ranked between two members of a couple for each of the hospitals. In what follows, we proceed to strengthen both the RF property and the CPC property to ensure the existence of a stable matching.

Remark 5 *Throughout this section we assume that couples' preferences satisfy the RF property and hospitals' preferences satisfy the CPC property.*

6.1 Condition on Couples' Preferences

In view of this, we further strengthen the RF property of couples' preferences by putting a restriction on the preferences of couples. We call this property Strong Responsive for F (SRF). A verbal description of this property is given below, which is followed by a formal definition.

Let $P^0_C = (\{P^0_d\}_{d\in D\setminus S}, \{P^0_C\}_{c\in C})$ be a given collection of preferences of non-single doctors (that is, the doctors in $D \setminus S$) and couples such that for all $c \in C$, P^0_c satisfies the RF property. Given such a collection of preferences P^0_C, an extension of P^0_C refers to any preference profile where (i) the preferences of non-single doctors and couples are given as P^0_C, and (ii) hospitals' preferences satisfy the CPC property.

In what follows, we present a condition, called the Strong Responsive for F (SRF) property, that we use in describing situations where stable matchings exist. The SRF property says the following. Consider a $c = \{f, m\} \in C$. Then P^0_c satisfies the SRF property if the following happens: for all hospitals h and h' in H, with h not being the top ranked hospital of f, if the couple prefers (h, h) to (h, h') and m prefers h' to h, then f prefers h' to h as well according to her individual preference. In other words, this property says that if m has to violate responsiveness to be together with f at hospital h, then this violation has to occur at the hospital which is preferred to h according to P^0_f. Now, we provide the formal definition below.

Definition **12** *A collection of preferences P^0_C satisfying the RF property is said to satisfy the Strong RF (SRF) property if for all $c = \{f, m\} \in C$ and all $h, h' \in H$ such that $r_1(P^0_f) \neq h$, $(h, h)P^0_c(h, h')$ and $h'P^0_m h$ implies $h'P^0_f h$.*

The following lemma says that the SRF property is the same as the RF property if there are only two hospitals.

Lemma **1** *Suppose $|H| = 2$. Then, a collection of preferences P^0_C of the couples satisfies the SRF property if and only if it satisfies the RF property.*

Proof: By the definition of the SRF property, if P^0_C satisfies the SRF property, then it satisfies the RF property. We prove the converse when there are two hospitals. Suppose $H = \{h_1, h_2\}$, and assume for contradiction that the preference P^0_c of a couple c satisfies the RF property but violates the SRF property. Without loss of generality, let us assume that $(h_1, h_1)P^0_c(h_1, h_2)$, $h_2 P^0_m h_1$, and $h_1 P^0_f h_2$. However, since the number of hospitals is 2, $h_1 P^0_f h_2$ implies $r_1(P^0_f) = h_1$, which is a contradiction to our assumption that the SRF is violated for P^0_c. ■

Now we present our theorem, which provides a necessary and sufficient condition for the existence of a stable matching at every possible extension of a given collection of preferences P^0_C. In particular, it says that a stable matching exists at every possible extension of P^0_C if and only if P^0_C satisfies the SRF property.

THEOREM **4** *(i) If P^0_C satisfies the SRF property, then a stable matching exists at every extension of P^0_C.*

(ii) If P^0_C does not satisfy the SRF property, then there always exists an extension of P^0_C at which there is no stable matching.

The proof of this theorem is relegated to Appendix.

6.2 Condition on Hospitals' preferences

Now, we further strengthen the CPC property of hospitals' preferences by putting a restriction on their preferences. We call this property Strong Common Preference over Couples (SCPC). A verbal description of this property is given below, which is followed by a formal definition.

Let $P^0_H = (\{P^0_h\}_{h \in H})$ be a given collection of preferences of hospitals, P^0_h satisfies the CPC property. Given such a collection of preferences P^0_H, an extension of P^0_H refers to any preference profile where (i) the preferences of couples satisfy the RF property, and (ii) hospitals' preferences are given in P^0_H.

In what follows, we present a condition, called Strong Common Preference over Couples (SCPC) property, that we use in describing situations where stable matchings exist. The SCPC property says that for each hospital and for any couple $c \in C$, the set of doctors who are preferred to its female member (that is, the more preferred member) remains the same for all hospitals. We now provide a formal definition of this property.

Definition **13** *A collection of CPC preferences P_H^0 satisfies the Strong Common Preferences over Couples (SCPC) property if for all $h, h' \in H$, all $c = \{f, m\} \in C$, and all $d \in D$, we have $dP_h^0 f$ if and only if $dP_{h'}^0 f$.*

Our next theorem says that the SCPC property is a sufficient condition on hospitals' preferences to guarantee the existence of a stable matching at its preference extension of P_H^0 where the couples' preferences satisfy the RF property.

Theorem **5** *If a collection of preferences of hospitals P_H^0 satisfies the SCPC property, then a stable matching exists at its every preference extension of P_H^0.*

7 Conclusion

In this paper, we have considered many-to-one matching problems between doctors and hospitals where doctors consist of some couples. First, we have considered the case where hospitals have a common preference over the individual doctors. We have shown that when a couple is allowed to violate responsiveness only for togetherness, a stable matching exists at every preference profile if and only if the lesser preferred member (according to the common preference of the hospitals) of the couple is ready to violate responsiveness to be together with the more preferred member. We have further provided necessary and sufficient conditions for the existence of a stable matching at every preference profile when a couple is allowed to violate responsiveness arbitrarily.

Next, we have considered the case where hospitals need not have a common preference over individual doctors. We have shown that under the common preference over couples' property of the hosptials, a stable matching exists if and only if the couples' preferences satisfy the SRF property. Moreover, we have further shown that if hospitals' preferences satisfy Strong CPC condition, then a stable matching always exists, when the couples' preferences satisfy the RF property.

An interesting open problem would be to consider the situation where (i) hospitals are partitioned based on geographical regions (that is, hospitals in the same geographical region are in one partition) and (ii) couples' preferences violate responsiveness in order for them to be employed at hospitals that are located in the same region. It follows from Theorem 2 that a stable matching will exist in this setting if hospitals preferences satisfy SCPI. However, SCPI need not be a necessary condition for the said existence. We leave the problem of finding the exact necessary and sufficient condition for the existence of a stable matching for future research.

Appendix: Remaining proofs

Proof of Theorem 1

Proof:[Part (i)] The proof of this part is constructive. Suppose P_C^0 satisfies the RF property. We show that every extension $\underset{\sim}{P}$ of P_C^0 has a stable matching.

Take an extension $\underset{\sim}{P}$ of P_C^0.

Recall that by our initial assumption on CPI, $m_i P_{hp}^0 m_j$ for all $i, j \in \{1, \ldots, k\}$ such that $i < j$. In the following, we present an algorithm that produces a stable matching at $\underset{\sim}{P}$.

Algorithm 1: This algorithm involves $k+1$ steps. We present the 1^{st} step and a general step of the algorithm.

Step 1: Use SDA to match all the doctors ranked above m_1 according to P_{hp}^0. Suppose f_1 is matched to some hospital h_1. Then, match m_1 using SDA where m_1 proposes according to the preference $P^0_{m_1|h_1}$.

$$\vdots$$

Step j: Having matched all the doctors from the top till m_{j-1} according to P^0_{hp} in steps 1 to $j-1$, use SDA to match all doctors ranked below m_{j-1} and above m_j according to P^0_{hp}. Suppose f_j is matched to some hospital h_j. Then, match m_j by SDA where m_j proposes according to the preference $P^0_{m_j|h_j}$.

$$\vdots$$

Continue this process till Step k and then match the remaining single doctors by SDA in step $k+1$.

Let μ be the outcome of Algorithm 1. We show that μ is stable at $\underset{\sim}{P}$.

First, we show that μ cannot be blocked by (h, s) for some $h \in H$ and $s \in S$. Assume for contradiction that some pair (h, s) blocks μ. By the nature of Algorithm 1, all doctors who propose before s are ranked above s in P^0_{hp}. Since $s \notin \mu(h)$, this means either $\mu(s)\underset{\sim}{P}_s h$ or $dP^0_{hp}s$ for all $d \in \mu(h)$ and $|\mu(h)| = \kappa_h$. Clearly, if $\mu(s)\underset{\sim}{P}_s h$ then s does not block with h. On the other hand, if $dP^0_{hp}s$ for all $d \in \mu(h)$ and $|\mu(h)| = \kappa_h$, then by responsiveness of hospitals' preferences, we have $\mu(h)\underset{\sim}{P}_h((\mu(h) \setminus d) \cup s)$ for all $d \in \mu(h)$. Therefore, h does not block with s. This proves that μ can not by blocked a hospital and a single doctor.

Now, we show that μ cannot be blocked by $((h_1, h_2), c)$ for some $h_1, h_2 \in H$ and $c \in C$. Assume for contradiction that some $((h_1, h_2), c)$ blocks μ. Let $c = \{f, m\}$. We complete the proof in two steps.

Step 1: In this step, we show that if $((h_1, h_2), c)$ blocks μ, then $((\mu(f), h_2), c)$ also blocks μ. Clearly, if $\mu(f) = h_1$, then there is nothing to show. So, suppose $\mu(f) \neq h_1$.

First, we claim $\mu(f)P^0_f h_1$. Assume for contradiction that $h_1 P^0_f \mu(f)$. Since f proposes according to P^0_f and all the doctors who propose before f are ranked above f in P^0_{hp}, $f \notin \mu(h_1)$ implies that $dP^0_{hp}f$ for all $d \in \mu(h_1)$ and $|\mu(h_1)| = \kappa_{h_1}$. By responsiveness of hospitals' preferences, this means $\mu(h_1)\underset{\sim}{P}_{h_1}((\mu(h_1) \setminus d) \cup f)$ for all $d \in \mu(h_1)$. However, this contradicts that $((h_1, h_2), c)$ blocks μ. Therefore, $\mu(f)P^0_f h_1$.

Next, we show that $(\mu(f), h_2)P^0_c(h_1, h_2)$. Assume for contradiction that $(h_1, h_2)P^0_c(\mu(f), h_2)$. If $h_1 \neq h_2$, then RVT implies $h_1 P^0_f \mu(f)$, which is a contradiction. On the other hand, if $h_1 = h_2$, then by the RF property implies $h_1 P^0_f \mu(f)$, which is a contradiction.

Now, we complete Step 1. Since $((h_1, h_2), c)$ blocks μ, it must be that $((\mu(h_2) \setminus d) \cup m)\tilde{P}_{h_2}\mu(h_2)$ for some $d \in \mu(h_2)$. Because $(\mu(f), h_2)P^0_c(h_1, h_2)$, it follows that $((\mu(f), h_2), c)$ blocks μ.

Step 2: In this step, we show that $((\mu(f), h_2), c)$ cannot block μ.

Suppose $\mu(f) = h$. Because $(\mu(f), h_2)P^0_c(\mu(f), \mu(m))$, the definition of $P^0_{m|h}$ implies $h_2 P^0_{m|h}\mu(m)$. Since all doctors who propose before m are ranked above m in P^0_{hp} and $m \notin \mu(h_2)$, it must be that $dP^0_{hp}m$ for all $d \in \mu(h_2)$ and $|\mu(h_2)| = \kappa_{h_2}$. By responsiveness of hospitals' preferences, this means $\mu(h_2)\underset{\sim}{P}_{h_2}((\mu(h_2) \setminus d) \cup m)$ for all $d \in \mu(h_2)$. However, this contradicts that $((\mu(f), h_2), c)$ blocks μ.

This completes the first part of the theorem.

[Part (ii)] Suppose P^0_C does not satisfy the RF property. We show that there is an extension of P^0_C with no stable matching. Since P^0_C does not satisfy the RF property, there must exist a couple $c = \{f, m\}$ and two hospitals $h_1, h_2 \in H$ such that $(h_1, h_1)P^0_c(h_2, h_1)$ and $h_2 P^0_f h_1$. Moreover, since $h_2 P^0_f h_1$, it follows from the definition of RVT that $(h_2, h_2)P^0_c(h_1, h_2)$.

Consider a profile $\underset{\sim}{P}$ such that

1. there are doctors $d_1, d_2 \in D \setminus \{f, m\}$ with $fP^0_{hp}d_1P^0_{hp}d_2P^0_{hp}m$ such that $r_1(\underset{\sim}{P}_{d_1}) = h_1$ and $r_1(\underset{\sim}{P}_{d_2}) = h_2$,

2. $|\{d : dP^0_{hp}f \text{ and } r_1(\underset{\sim}{P}_d) = h_2\}| = \kappa_{h_2} - 2$, $|\{d : dP^0_{hp}f \text{ and } r_1(\underset{\sim}{P}_d) = h_1\}| = \kappa_{h_1} - 2$, and $|\{d : dP^0_{hp}f \text{ and } r_1(\underset{\sim}{P}_d) = h\}| = \kappa_h$ for all $h \neq h_1, h_2$, and

3. the preferences of all couples other than c satisfy responsiveness.

Since $\sum_{h\in H}\kappa_h = |D|$ by the construction of $\underset{\sim}{P}$, the four bottom-ranked (lowest ranked) doctors in P^0_{hp} are f, d_1, d_2, m. We show that there is no stable matching at $\underset{\sim}{P}$. Assume for contradiction that a matching μ is stable matching at $\underset{\sim}{P}$. Since μ is stable at $\underset{\sim}{P}$, by the construction $\underset{\sim}{P}$, it is straight forward that $\mu(d) = r_1(\underset{\sim}{P}_d)$ for all $dP^0_{hp}f$.

Because $|\{d : dP^0_{hp}f \text{ and } r_1(\underset{\sim}{P}_d) = h\}| = \kappa_h$ for all $h \neq h_1, h_2$, stability of μ implies that the doctors f, d_1, d_2, m cannot be matched to any hospital other than h_1 and h_2. Moreover, since $|\{d : dP^0_{hp}f \text{ and } r_1(\underset{\sim}{P}_d) = h_2\}| = \kappa_{h_2} - 2$ and $|\{d : dP^0_{hp}f \text{ and } r_1(\underset{\sim}{P}_d) = h_1\}| = \kappa_{h_1} - 2$, exactly two doctors among f, d_1, d_2, m must be matched to each of h_1 and h_2.

Now, we distinguish the following cases depending on the allocation of the couple c and show that μ is not stable in any of these cases.

- Suppose $\mu(c) = (h_2, h_2)$.

  Then, (h_2, d_2) blocks μ as $r_1(\underset{\sim}{P}_{d_2}) = h_2$ and $d_2P^0_{hp}m$.

- Suppose $\mu(c) = (h_1, h_2)$.

  Then, $((h_2, h_2), c)$ blocks μ as $fP^0_{hp}d_1P^0_{hp}d_2$, and by the definition of RVT $(h_2, h_2)P^0_c(h_1, h_2)$.

- Suppose $\mu(c) = (h_1, h_1)$.

  Then, (h_1, d_1) blocks μ as $r_1(\underset{\sim}{P}_{d_1}) = h_1$ and $d_1P^0_{hp}m$.

- Suppose $\mu(c) = (h_2, h_1)$.

  Then, $((h_1, h_1), c)$ blocks μ as $fP^0_{hp}d_1P^0_{hp}d_2$, and by the initial assumption, $(h_1, h_1)P^0_c(h_2, h_1)$.

This completes the proof. ■

Proof of Theorem 2

Proof:Suppose P^0_{hp} satisfies SCPI. We show that there exists a stable matching for every unrestricted extension $\underset{\sim}{P}$ of P^0_{hp}. Let us partition the set of couples C into two subsets C_1 and C_2 such that for all $c = \{f, m\} \in C_1$, $|\{d \in D : dP^0_{hp}m\}| < \kappa_h$, and for all $c = \{f, m\} \in C_2$, $|\{d \in D : dP^0_{hp}m\}| \geq \kappa_h$. Let us index the couples in C_2 as $\{f^1, m^1\}, \ldots \{f^l, m^l\}$ where $m^i P^0_{hp} m^j$ for all $i, j \in \{1, \ldots, l\}$ with $i < j$. Since P^0_{hp} satisfies SCPI and $|\{d \in D : dP^0_{hp}m\}| \geq \kappa_h$ for all $c = \{f, m\} \in C_2$, this implies $f^i P^0_{hp} f^j$ for all $i, j \in \{1, \ldots, l\}$ with $i < j$. In the following, we present an algorithm that produces a stable matching at $\underset{\sim}{P}$. Clearly, by construction, for any $c = \{f, m\} \in C_1$ and $c' = \{f', m'\} \in C_2$, we have $mP^0_{hp}f'$.

Algorithm 2: We present the 1^{st} step and a general step of the algorithm. **Step 1**: Use SDA to match all the doctors who are ranked above f^1 according to P^0_{hp} in the following manner. All the single doctors $s \in S$ propose according to $\underset{\sim}{P}_s$. For any couple $c = \{f, m\} \in C_1$, f proposes according to her conditional preference in $\tilde{P}_c$. More formally, f first proposes to the hospital h_f such that (h_f, h_m) appears at the top position of $\tilde{P}_c$ for some hospital h_m. If f is rejected by h_f, she proposes to the hospital h'_f that appears after h_f in the f-component of the preference $\tilde{P}_c$. In other words, h'_f is such that there is no hospital h''_f other than h_f such that (h''_f, h''_m) appears above (h'_f, h'_m) for some h'_m and h''_m according to the preference $\tilde{P}_c$. f continues to propose this way till she is matched. Once f is matched with some hospital h, m starts proposing to the hospitals according to the preference $\tilde{P}_{m|h}$ till he is matched. Once all the doctors who are ranked above f^1 are matched, c^1 proposes to the hospitals according to the preference $\underset{\sim}{P}_{c_1}$ till both members of c^1 get matched. More formally, c^1 first proposes to $r_1(\underset{\sim}{P}_{c^1})$, and if any member of the couple is rejected, then it proposes to $r_2(\underset{\sim}{P}_{c^1})$, and so on until both members of the couple are accepted by the corresponding hospitals.

$\vdots$

Step j: Having matched all the doctors from the top till m^{j-1} in P^0_{hp} in steps

1 to $j-1$, use SDA to match all the doctors that ranked below m^{j-1} and above r^j according to P^0_{hp}. Note that for $j > 1$, there is no couple $c = \{f, m\} \in C_1$ such that f or m is ranked below m^{j-1} and above f^j according to P^0_{hp}. For the couple, c^j proposes according to the preference $\underset{\sim}{P}_{c^j}$ till both of them are accepted by the corresponding hospitals. That is, c^j first proposes to $r_1(\underset{\sim}{P}_{c^j})$, and if at least one member of the couple is rejected, then they propose to $r_2(\underset{\sim}{P}_{c^j})$, and so on, till both of them are accepted.

$$\vdots$$

Continue this process till Step $l-1$. Having matched all the doctors from the top till m^{l-1} according to P^0_{hp} matched in steps 1 to $l-1$, use SDA to match all the doctors that are ranked below m^{l-1} and above f^l according to P^0_{hp}. We distinguish the following two cases to match remaining doctors. Note here, that the remaining doctors now include f^l and all the doctors ranked below f^l.

Case 1. Suppose there is no single doctor in between f^l and m^l in P^0_{hp} for all $h \in H$. Let c^l propose to $r_1(\underset{\sim}{P}_{c^k})$. If at least one member of the couple is rejected, then let c^l propose to $r_2(\underset{\sim}{P}_{c^l})$, and so on. Continue this process till both members of the couple are accepted. Finally, match all the remaining doctors using SDA.

Case 2. Suppose there is a single doctor, say s', in between f^l and m^l in P^0_{hp}. Note that by SCPI, there cannot be more than one single doctor in between f^l and m^l. Suppose H' is the set of hospitals that have at least one remaining vacancy. Let h' be the worst hospital in H' according to $\underset{\sim}{P}_{s'}$ and let $h \in H'$ be such that $(h, h')\underset{\sim}{R}_{c^l}(h'', h')$ for all $h'' \in H'$. Match c^l with (h, h') and s' to the hospital that has a remaining vacancy.

Let μ be the outcome of Algorithm 2. We show that μ is stable at $\underset{\sim}{P}$.

Assume for contradiction that μ is blocked by a hospital and a single doctor or a pair of hospitals and a couple. We complete the proof by considering the two cases of Algorithm 2 separately.

Case 1. Suppose Case 1 of Algorithm 2 holds.

First, we show that μ cannot be blocked by (h, s) for some $h \in H$ and $s \in S$. Assume for contradiction that some pair (h, s) blocks μ. By the nature of Algorithm 2, all the doctors who propose before s are ranked above s according to the SCPI P^0_{hp}. Moreover, for any $c = \{f, m\} \in C_2$, if $fP^0_{hp}s$, then by SCPI, $mP^0_{hp}s$. Since $s \notin \mu(h)$, by the nature of Algorithm 2, we have either $\mu(s)\underset{\sim}{P}_s h$ or $dP^0_{hp}s$ for all $d \in \mu(h)$ and $|\mu(h)| = \kappa_h$. Clearly, if $\mu(s)\underset{\sim}{P}_s h$ then s does not block with hospital h. On the other hand, if $dP^0_{hp}s$ for all $d \in \mu(h)$ and $|\mu(h)| = \kappa_h$, then by responsiveness of hospitals' preferences, we have $\mu(h)\underset{\sim}{P}_h(\mu(h) \setminus d) \cup s$ for all $d \in \mu(h)$. Therefore, hospital h does not block with s. This contradicts that (h, s) blocks μ.

Next we show that μ can not be blocked by $((h_1, h_2), c)$ for some $h_1, h_2 \in H$ and $c = \{f, m\} \in C$. First, we claim $c \notin C_1$. Note that if each hospital has enough vacancies to accommodate the couple together with all doctors who are ranked above it, c would not get rejected by any pair of hospitals it applies to and thus, it would have been matched to their top ranked pair of hospitals, which contradicts our assumption that c blocks μ. Therefore, it must not be the case that $|\{d \in D : dP^0_{hp}m\} < \kappa_h$ for all h, which means $c \notin C_1$.

In view of the preceding claim, it follows that if μ is blocked by $((h_1, h_2), c)$, then $c \in C_2$. By the nature of Algorithm 2, couple c proposes to (h_1, h_2) before proposing to $(\mu(f), \mu(m))$, and some hospital, say $h_i \in \{h_1, h_2\}$, rejects the corresponding member of the couple c. We distinguish the following two sub-cases.

Case 1.1. Suppose $h_1 \neq h_2$. Since h_i rejects a doctor from couple c, it must be that h_i has no vacancies when c proposes to (h_1, h_2). Since c is in C_2, we have that f and m are adjacent in P^0_{hp}. It follows that all the doctors in $\mu(h_i)$ are preferred to both f and m. Therefore, h_i will be worse off by removing a doctor from $\mu(h_i)$ and taking a member from the couple c, which contradicts that $((h_1, h_2), c)$ blocks μ.

Case 1.2. Suppose $h_1 = h_2$. Because h_1 rejects at least one member of c, it must be that h_1 has less than two vacancies when c proposes to (h_1, h_1). Let

D' be the set of doctors that are present in h_1 at the time when c makes the proposal to (h_1, h_1). By SCPI, the definition of C_2, and the nature Algorithm 2, this implies that each doctor in D' is preferred to both the doctors of the couple c. Again, by Algorithm 2, it follows that $D' \subseteq \mu(h_1)$. This means h_1 must release some doctors from D' in order to block with c. Since hospitals' preferences over sets of individuals satisfy responsiveness, therefore, h_1 will be worse off by removing two doctors from D' in order to take the couple. This contradicts that $((h_1, h_1), c)$ blocks μ.

This completes the proof of Theorem 2 for Case 1.

Case 2. Suppose Case 2 of Algorithm 2 holds. Note that after matching all the doctors from the top till f^l in P^0_{hp}, we have exactly three vacancies left since $\sum_{h\in H} \kappa_h = |D|$. Recall from Case 2 of our Algorithm that H' is the set of hospitals with at least one vacancy left, after all the doctors ranked above f^l have been matched.

By similar argument as in Case 1, (i) μ cannot be blocked by (h, s) for any $sP^0_{hp}f^l$, and (ii) μ cannot be blocked by $((h_1, h_2), c)$ for any c such that $c \neq c^l$.

First, we show μ cannot be blocked by (h, s'), where s' is the unique single doctor ranked between f^l and m^l in P^0_{hp}. Suppose not. Since $dP^0_{hp}s'$ for all $d \neq m^l$, it follows that $h \in H'$. By Algorithm 2, $\mu(s') \in H'$ and $\mu(m^l)$ is the worst hospital in H' according to $\underset{\sim}{P}_{s'}$. Since μ is blocked by (h, s'), we have $h\underset{\sim}{P}_{s'}\mu(s')\underset{\sim}{R}_{s'}\mu(m^l)$, which implies $h \neq \mu(m^l)$. We now show that $h \neq \mu(f^l)$. Assume for contradiction, $h = \mu(f^l)$. By our earlier argument, since $h \neq \mu(m^l)$, $\mu(f^l) = \mu(m^l)$ implies $h \neq \mu(f^l)$. Suppose $\mu(f^l) \neq \mu(m^l)$. This means all the doctors in $\mu(f^l)$ are ranked above s' according to P^0_{hp}, contradicting the fact that $\mu(f^l)$ and s' block μ. This shows $h \neq \mu(f^l)$. By the definition of Algorithm 2, $h \in \{\mu(s'), \mu(f^l), \mu(m^l)\}$. Since $h \notin \{\mu(f^l), \mu(m^l)\}$, it must be that $h = \mu(s')$, and hence h and s' can not block.

Now, we show that μ cannot be blocked by $((h_1, h_2), c^l)$ for some $h_1, h_2 \in H$. Since $dP^0_{hp}f^l$ for all $d \notin \{s', m^l\}$, it follows from Algorithm 2 and the definition of H' that $h_1, h_2 \in H'$. We complete the proof by distinguishing the following two cases.

Case 2.1. Suppose $h_2 = \mu(m^l)$. By Algorithm 2, $\mu(c^l)\underset{\sim}{R}_{c^l}(h_1, h_2)$ for all $h_1 \in H'$. Therefore, c^l will not block with (h_1, h_2).

Case 2.2. Suppose $h_2 \neq \mu(m^l)$. By Algorithm 2, this means all the doctors in h_2 are preferred to m^l according to P^0_{hp}. Therefore, h_2 will not block with m^l.

This completes the proof of Theorem 2 for Case 2. Since Case 1 and Case 2 are exhaustive, this completes the proof of Theorem 2. ■

Proof of Theorem 3

Proof:Suppose a CPI P^0_{hp} does not satisfy SCPI. We show that there exists an RVT extension of P^0_{hp} with no stable matching. Since P^0_{hp} does not satisfy SCPI, one of the following two cases must happen:

Case 1. There is a couple $c = \{f, m\}$ such that $r(m, P^0_{hp}) \neq |D|$, $|\{d \in D : fP^0_{hp}dP^0_{hp}m\}| > 0$ and $|\{d \in D : dP^0_{hp}m\}| \geq \kappa_h$ for some $h \in H$. Thus, there exist doctors d_1, d_2 such that $fP^0_{hp}d_1P^0_{hp}mP^0_{hp}d_2$ and a hospital h_1 such that $|\{d \in D : dP^0_{hp}m\}| \geq \kappa_{h_1}$.

Case 2. There is a couple $c = \{f, m\}$ such that $r(m, P^0_{hp}) = |D|$ and $|\{d \in D : fP^0_{hp}dP^0_{hp}m\}| > 1$. In other words, there exist doctors d_1, d_2 such that $fP^0_{hp}d_1P^0_{hp}d_2P^0_{hp}m$.

In the following, we present an RVT extension of P^0_{hp} with no stable matching for both Case 1 and Case 2.

Take hospitals $h_1, h_2 \in H$ and consider a preference profile $\underset{\sim}{P}$ such that

1. $r_1(\underset{\sim}{P}_f) = r_1(\underset{\sim}{P}_{d_2}) = h_2$ and $r_1(\underset{\sim}{P}_m) = r_1(\underset{\sim}{P}_{d_1}) = h_1$,

2. $r_2(\underset{\sim}{P}_f) = r_2(\underset{\sim}{P}_{d_2}) = h_1$ and $r_2(\underset{\sim}{P}_m) = r_2(\underset{\sim}{P}_{d_1}) = h_2$,

3. $(h_1, h_1)\underset{\sim}{P}_c(h_2, h_1)$ and $(h_1, h_1)\underset{\sim}{P}_c(h_2, h_2)$,

4. $(h_1, h_2)\underset{\sim}{P}_c(h, h')$ for all $h, h' \in H$ such that (h, h') does not belong to the set $\{(h_1, h_1), (h_2, h_2), (h_2, h_1), (h_1, h_2)\}$,

5. preference $\underset{\sim}{P}_c$ satisfies responsiveness for all pairs of hospitals other than (h_1, h_1),

6. preferences of all couples other than c satisfy responsiveness,

7. $|\{d : r_1(\underset{\sim}{P}_d) = h_1 \text{ and } dP^0_{hp}m\}| = \kappa_{h_1} - 1$, and

8. for all $d \notin \{f, m, d_1, d_2\}$, $|\{d : r_1(\underset{\sim}{P}_d) = h_2\}| = \kappa_{h_2} - 2$ and $|\{d : r_1(\underset{\sim}{P}_d) = h\}| = \kappa_h$ for all $h \neq h_1, h_2$.

Note that the assumption made in condition 7 is possible as $|\{d \in D : dP^0_{hp}m\}| \geq \kappa_{h_1}$ and $fP^0_{hp}m$. However, $r_1(\underset{\sim}{P}_f) \neq h_1$. We show that there is no stable matching at $\underset{\sim}{P}$ for both Case 1 and Case 2. Assume for contradiction that μ is a stable matching at $\underset{\sim}{P}$. Note that by the construction of $\underset{\sim}{P}$, for all doctors d such that $dP^0_{hp}f$, we must have $\mu(d) = r_1(\underset{\sim}{P}_d)$. In the following claim, we show that $\mu(d) \in \{h_1, h_2\}$ for all $d \in \{f, m, d_1, d_2\}$.

Claim **1** *For all $d \in \{f, m, d_1, d_2\}$, $\mu(d) \in \{h_1, h_2\}$.*

Proof: First, we show $\mu(d) \in \{h_1, h_2\}$ for $d \in \{f, m\}$. Suppose $\mu(d) = h'$ for some $d \in \{f, m\}$ and some $h' \notin \{h_1, h_2\}$. We complete the proof for the case where $\mu(m) = h'$, the same for the case $\mu(f) = h'$ follows from similar arguments. Let $\mu(c) = (h, h')$ for some $h \in H$. Consider the matchings (h, h_1) and (h, h_2) of the couple c. Note that by responsiveness, $(h, h_1)\underset{\sim}{P}_c(h, h')$ and $(h, h_2)\underset{\sim}{P}_c(h, h')$. Further, since $\sum_{h \in H} \kappa_h = |D|$ and $\mu(d) = r_1(\underset{\sim}{P}_d)$ for all doctors d such that $dP^0_{hp}m$, $\mu(m) = h'$ implies that there must be a doctor d' with $mP^0_{hp}d'$ such that either $d' \in \mu(h_1)$ or $d' \in \mu(h_2)$.

Suppose not, then points 7. and 8. imply that there exists a doctor $d'' \notin \{f, m, d_1, d_2\}$ such that $r_1(\underset{\sim}{P}_{d''}) \neq h_1$ and $d'' \in \mu(h_1)$ or $r_1(\underset{\sim}{P}_{d''}) \neq h_2$ and $d'' \in \mu(h_2)$. Thus, d'' can block μ with $r_1(\underset{\sim}{P}_d)$. If not, then by a recursive argument as above, by points 7. and 8.there exists another doctor preferred to d'' such that she is in not in her most preferred hospital and is matched to $r_1(\underset{\sim}{P}_d)$. Continuing like this, we get that there exists a doctor not in $\{f, m, d_1, d_2\}$ such that she is not in her most preferred hospital and can thus block μ with that hospital.

However, if there exists a doctor d' with $mP^0_{hp}d'$ such that either $d' \in \mu(h_1)$ or $d' \in \mu(h_2)$ then, the couple c blocks μ with either (h, h_1) or (h, h_2) contradicting the stability of μ. Therefore, $\mu(d) \in \{h_1, h_2\}$ for all $d \in \{f, m\}$.

Now, we show $\mu(d) \in \{h_1, h_2\}$ for $d \in \{d_1, d_2\}$. Suppose $\mu(d) = h'$ for some $d \in \{d_1, d_2\}$ and some $h' \notin \{h_1, h_2\}$. Since $\mu(d) \in \{h_1, h_2\}$ for all $d \in \{f, m\}$ and $\mu(d) = r_1(\underset{\sim}{P}_d)$ for all doctors d such that $dP^0_{hp}f$, there must be a doctor d' with $d_2P^0_{hp}d'$ such that either $d' \in \mu(h_1)$ or $d' \in \mu(h_2)$. Because $r_k(\underset{\sim}{P}_d) \in \{h_1, h_2\}$ for all $k = 1, 2$, if $\mu(d') = h_1$, then d blocks μ with h_1, and if $\mu(d') = h_2$, then d blocks μ with h_2. This contradicts the stability of μ. Therefore, $\mu(d) \in \{h_1, h_2\}$ for all $d \in \{d_1, d_2\}$.

This completes the proof of Claim 1. $\square$

Now, we distinguish the following cases depending on the allocation of couple c and show that μ is not stable for each of these cases.

- Suppose $\mu(c) = (h_1, h_1)$.

  Since $|\{d : r_1(\underset{\sim}{P}_d) = h_1 \text{ and } dP^0_{hp}f\}| = \kappa_{h_1} - 2$ and $fP^0_{hp}d_1$, thus $d_1 \notin \mu(h_1)$. Because $h_1\underset{\sim}{P}_{d_1}h_2$ and $d_1P^0_{hp}m$, this means (h_1, d_1) blocks μ.

- Suppose $\mu(c) = (h_2, h_1)$.

  Then, $((h_1, h_1), c)$ blocks μ as $fP^0_{hp}d_1P^0_{hp}d_2$ and $(h_1, h_1)\underset{\sim}{P}_c(h_2, h_1)$.

- Suppose $\mu(c) = (h_2, h_2)$.

  By the construction of $\underset{\sim}{P}$, $(h_1, h_1)\underset{\sim}{P}_c(h_2, h_2)$ and $h_2\underset{\sim}{P}_{d_2}h_1$. If Case 1 holds, then $((h_1, h_1), c)$ blocks μ as $fP^0_{hp}d_1$ and $mP^0_{hp}d_2$. On the other hand, if Case 2 holds, then (h_2, d_2) blocks μ as $d_2P^0_{hp}m$.

- Suppose $\mu(c) = (h_1, h_2)$.

  Since $h_2\underset{\sim}{P}_f h_1$, by RVT, $(h_2, h_2)\underset{\sim}{P}_c(h_1, h_2)$. This, together with the fact that $fP^0_{hp}d_1P^0_{hp}d_2$, means μ is blocked by $((h_2, h_2), c)$.

This completes the proof of Theorem 3. ■

Proof of Theorem 4

Proof: [Part (i)] The proof of this part is constructive. Suppose P^0_C satisfies the SRF property. We show that every extension of $\underset{\sim}{P}$ of P^0_C has a stable matching.

Take an extension $\underset{\sim}{P}$ of P^0_C. We present an algorithm which produces a stable matching at $\underset{\sim}{P}$.

Algorithm 3: Use DPDA, where at each stage, each single doctor s proposes according to $\underset{\sim}{P}_s$, and for any couple $c = \{f, m\}$, f proposes according to P^0_f, and m, if not already matched, proposes according to $P^0_{m|h}$, where h is the hospital f proposes to.

Let μ be the outcome of this algorithm. We make a remark that we repeatedly refer in our proof.

Remark **6** *If a doctor d is rejected by some hospital h at some stage of this algorithm, then (h, d) cannot block μ. This is because, by DPDA, a hospital only rejects a doctor d if $d' \underset{\sim}{P}_h d$ for all $d' \in \mu(h)$.*

First, we show that μ cannot be blocked by (h, s) for some $h \in H$ and $s \in S$. Assume for contradiction that some pair (h, s) blocks μ. Since $s \notin \mu(h)$, this means either $\mu(s) \underset{\sim}{P}_s h$ or s was rejected by h at some stage of the algorithm. Clearly, if $\mu(s) \underset{\sim}{P}_s h$ then s does not block with h. On the other hand, if s had proposed to h and was rejected by h at an earlier stage, then by the above Lemma 6, (h, s) cannot block μ.

Now, we show that μ cannot be blocked by $((h_1, h_2), c)$ for some $h_1, h_2 \in H$ and $c \in C$. Assume for contradiction that $((h_1, h_2), c)$ blocks μ for some $h_1, h_2 \in H$ and $c \in C$. We distinguish the following two cases.

Case 1. Suppose $\mu(f) = r_1(P^0_f) = h_f$.

First, we show that $\mu(m) \neq h_2$. To the contrary, suppose $\mu(m) = h_2$. Then, $(h_1, h_2) P^0_c (h_f, h_2)$ implies $h_1 P^0_f h_f$, which contradicts the fact that $h_f =$

$r_1(P_f^0)$.

Next, we show that $(h_f, h_2)P_c^0(h_f, \mu(m))$. Since $((h_1, h_2), c)$ blocks μ and $\mu(f) = h_f$, $(h_1, h_2)P_c^0(h_f, \mu(m))$. If $h_1 = h_f$, there is nothing to prove. Suppose $h_1 \neq h_f$. Then, by the responsiveness with respect to f, we have $(h_f, h_2)P_c^0(h_1, h_2)$. Since $(h_1, h_2)P_c^0(h_f, \mu(m))$ and $\mu(m) \neq h_2$, this implies $(h_f, h_2)P_c^0(h_f, \mu(m))$. Since $((h_1, h_2), c)$ blocks μ and $\mu(f) = h_f$, it follows that $((h_f, h_2), c)$ also blocks μ.

By the definition of $P_{m|h_f}^0$, $(h_f, h_2)P_c^0(h_f, \mu(m))$ implies $h_2 P_{m|h_f}^0 \mu(m)$. Therefore, by the definition of Algorithm 3, it must be that m had proposed to h_2 and got rejected at an earlier stage of Algorithm 3. Hence, by the definition of DPDA, $d' \underset{\sim}{P}_{h_2} m$ for all $d' \in \mu(h_2)$. Thus $((h_f, h_2), c)$ cannot block μ. This completes the proof for Case 1.

Case 2. Suppose $\mu(f) \neq h_f$.

We first prove the following lemma.

LEMMA **2** *If $\mu(f) \neq h_f$, then $\mu(f) R_f^0 h_1$.*

Proof: Assume for contradiction that $h_1 P_f^0 \mu(f)$. Since by Algorithm 3, f proposes according to P_f^0, this implies that f had proposed to h_1 and got rejected. By the nature of DPDA, this implies $d \underset{\sim}{P}_{h_1} f$ for all $d \in \mu(h_1)$. Therefore, $((h_1, h_2), c)$ cannot block μ, a contradiction. $\square$

LEMMA **3** *If $h_1 = h_2 = h$, then $((h, h), c)$ cannot block μ.*

Proof: First, we show that $h P_m^0 \mu(m)$. Suppose not. By the SRF property, $(h, h)P_c^0(\mu(f), \mu(m))$ implies $h P_f^0 \mu(f)$. Therefore, it must be that f had proposed to h in DPDA and got rejected, and hence $d \underset{\sim}{P}_h f$ for all $d \in \mu(h)$. This contradicts our assumption that $((h, h), c)$ blocks μ.

From Lemma 2, we know that $\mu(f) R_f^0 h$. Since the couples' preferences satisfy responsiveness with respect to f, we have $(\mu(f), h)R_c^0(h, h)$. Since $((h, h), c)$ blocks μ, we have $(h, h)P_c^0(\mu(f), \mu(m))$. Thus, from the above argument, we have $(\mu(f), h)P_c^0(\mu(f), \mu(m))$ which implies $h P_{m|h}^0 \mu(m)$. By the nature of the algorithm, this implies that m had proposed to h and got rejected. Thus for all $d \in \mu(h)$, $d\underset{\sim}{P}_h m$. Thus $((h,h),c)$ can not block μ. $\square$

Therefore, $h_1 \neq h_2$. Now, we know that couples only violate responsiveness for togetherness. Therefore $(h_1,h_2)P_c^0(\mu(f),\mu(m))$ and Lemma 2 imply $h_2 P_m^0 \mu(m)$. Since P_c^0 follows responsiveness for f, Lemma 2 implies $(\mu(f),h_2)R_c^0(h_1,h_2)$. Since $((h_1,h_2),c)$ blocks μ, $(h_1,h_2)P_c^0(\mu(f),\mu(m))$. This in turn means $(\mu(f),h_2)P_c^0(\mu(f),\mu(m))$. Therefore, $h_2 P^0_{m|\mu(f)}\mu(m)$. However, this implies that m had proposed to h_2 and got rejected in some previous round of Algorithm 3, and hence, $d\underset{\sim}{P}_{h_2} m$ for all $d \in \mu(h_2)$. Therefore, $((h_1,h_2),c)$ cannot block μ.

Since Case 1 and Case 2 are exhaustive, this completes the proof.

[Part (ii)] Suppose P_C^0 does not satisfy the SRF property. We show that there exists an extension of P_C^0 with no stable matching.

Since P_C^0 does not satisfy the SRF property, there exists a couple $c = \{f,m\} \in C$ and hospitals $h_1,h_2,h_3 \in H$ such that $r_1(P_f^0) = h_1 \neq h_3$. Further, $(h_3,h_3)P_c^0(h_3,h_2)$ and $h_2 P_m^0 h_3$ but we have $h_3 P_f^0 h_2$. This yields $h_1 P_f^0 h_3 P_f^0 h_2$, which in turn implies $|H| \geq 3$. Since $\kappa_h \geq 2$ for all $h \in H$, there exist at least four doctors $\{d_1,d_2,d_3,d_4\} \in D \setminus \{f,m\}$ and let us denote the set of doctors $\{f,m,d_1,d_2,d_3,d_4\}$ by D_1. Consider a preference profile $\underset{\sim}{P}$ such that

1. for all $h \in H \setminus \{h_1,h_2,h_3\}$, $|\{d : d\underset{\sim}{P}_h d' \text{ and } r_1(\underset{\sim}{P}_d) = h\}| = \kappa_h$ for $d' \in \{f,m,d_1,d_2,d_3,d_4\}$,

2. for all $h \in \{h_1,h_2,h_3\}$, $|\{d : d\underset{\sim}{P}_h d' \text{ and } r_1(\underset{\sim}{P}_d) = h\}| = \kappa_h - 2$ for $d' \in \{f,m,d_1,d_2,d_3,d_4\}$,

3. $P_{c'}^0$ satisfies responsiveness for all couples $c' \in C \setminus \{c\}$, and

4. The preferences of h_1,h_2,h_3 over $\{f,m,d_1,d_2,d_3,d_4\}$ and preferences of $\{f,m,d_1,d_2,d_3,d_4\}$ over h_1,h_2,h_3 is given by Table 3.

Lemma **4** *If a matching μ is stable at $\underset{\sim}{P}$, then $\mu(d) = r_1(\underset{\sim}{P}_d)$ for all $d \notin \{f,m,d_1,d_2,d_3,d_4\}$.*

Proof: Suppose not. Then, there exists a doctor $d \notin \{f, m, d_1, d_2, d_3, d_4\}$, such that $r_1(\utilde{P}_d) = h$, but $d \notin \mu(h)$. Take any doctor $d \notin \{f, m, d_1, d_2, d_3, d_4\}$ and let $h = r_1(\utilde{P}_d)$. By the construction of the preference profile $\utilde{P}$, we have $|\{d' : d'\utilde{P}_h d \text{ and } r_1(\utilde{P}_{d'}) = h\}| < \kappa_h$. Therefore, there exists $d' \in \mu(h)$ and $d' \notin \{f, m, d_1, d_2, d_3, d_4\}$ with $r_1(\utilde{P}_{d'}) \neq h$ such that either $d\utilde{P}_h d'$ or $d'\utilde{P}_h d$. Clearly, if $d\utilde{P}_h d'$, then (h, d) blocks μ, a contradiction.

Suppose $d'\utilde{P}_h d$ and $r_1(\utilde{P}_{d'}) \neq h$. Let $h' = r_1(d')$. Since $r_1(d') = h'$ and (h', d') does not block μ (by the assumption that μ is stable), it follows by using the same argument as in preceding paragraph that there exists a pair (h'', d'') such that $h'' = r_1(\utilde{P}_{d''})$, $d'' \notin \mu(h'')$, $d'' \notin \{f, m, d_1, d_2, d_3, d_4\}$, and there exists a doctor $\hat{d} \in \mu(h'')$ such that $\hat{d}\utilde{P}_{h''}d''$.

Continuing in this manner, by means of Part (1) and Part (2) of the definition of the preference profile $\utilde{P}$, we get hold of a pair (h^*, d^*) such that $h^* = r_1(\utilde{P}_{d^*})$, $d^* \notin \mu(h^*)$, $d^* \notin \{f, m, d_1, d_2, d_3, d_4\}$, and $d^*\utilde{P}_{h^*}\bar{d}$ for some $\bar{d} \in \mu(h^*)$. However, this implies that μ is blocked by (h^*, d^*), a contradiction.
$\square$

Since $\mu(d) = r_1(\utilde{P}_d)$ for all $d \notin \{f, m, d_1, d_2, d_3, d_4\}$, by the definition of the preference profile $\utilde{P}$, we can restrict our attention to the scenario presented in Example 3. However, we have already argued the there is no stable matching for the scenario in Example 3. Therefore, μ cannot be stable.
$\blacksquare$

Proof of Theorem 5

Proof: The proof of this theorem is constructive. Suppose P^0_H satisfies SCPC property. We show that every extension $\utilde{P}$ of P^0_H has a stable matching.

Consider an extension $\utilde{P}$ of P^0_H. Since P^0_C satisfies SCPC, therefore for any couple $c = \{f, m\} \in C$ and all hospitals $h, h' \in H$, $\{d : d \in D \text{ and } dP^0_h f\} = \{d : d \in D \text{ and } ddP^0_{h'} f\}$. That is the set of doctors preferred to f under a hospital's preference is the same for of all the hospitals in H. Without loss of generality, $C = \{\{f_1, m_1\}, \dots \{f_k, m_k\}\}$ where $f_i P^0_h f_j$ for all $i < j \in \{1, \dots, k\}$ and all $h \in H$.

For any hospital $h \in H$, let $F^h_i = \{d : f_{i-1} P^0_h d \text{ and } d R^0_h f_i\}$ for all $i \in \{1, \dots, k\}$. In other words, F^h_i is the collection of doctors who are weakly preferred to f_i and strictly less preferred to f_{i-1} according to P^0_h. By the definition of SCPC, it follows that $F^h_i = F^{h'}_i$ for all $h, h' \in H$. In view of this, let us denote F^h_i by F_i, that is, let us drop the superscript h.

Now we present an algorithm that produces a stable matching at $\underset{\sim}{P}$.

ALGORITHM 4: This algorithm involves $k+1$ steps. We present the 1^{st} step and a general step of the algorithm.

Step 1: Use DPDA to match all the doctors in F_1 where all the single doctors $s \in S$ propose according to $\underset{\sim}{P}_s$ and f_1 proposes according to $\underset{\sim}{P}_{f_1}$. Let f_1 be matched to hospital h_1.

$\vdots$

Step j: After matching all the doctors who are ranked above and including f_{j-1} in steps 1 to $j-1$, use DPDA to match all the doctors in F_j, where all the singles $s \in S$ propose according to $\underset{\sim}{P}_s$ and f_j proposes according to $\underset{\sim}{P}_{f_j}$. Match each (if any) $m_i \in F_j$ by using DPDA according to $\underset{\sim}{P}_{m|h_i}$, where h_i is the hospital where f_i is matched to.

$\vdots$

Continue this process till Step k. Having matched all the doctors who are ranked above and including f_k, we proceed to match the remaining doctors in the following manner. Match each single doctor $s \in S$ using DPDA according to $\underset{\sim}{P}_s$, and, as before, match each m_i in the set of remaining doctors by using DPDA according to $\underset{\sim}{P}_{m|h_i}$, where h_i is the hospital where f_i is matched to.

Let μ be the outcome of Algorithm 4. We show that μ is stable at $\underset{\sim}{P}$.

By using similar arguments as in Lemma 6, and the fact that we use DPDA at every stage to match the hospitals, it follows that if a doctor d is rejected by hospital h at some stage of the algorithm, then (h, d) can not block μ. This in particular means that a hospital and a single doctor cannot block μ. This is because, if a single doctor s prefers a hospital h to $\mu(s)$, then by the definition of DPDA, he/she must have already proposed to the hospital h and got rejected before getting matched with $\mu(s)$.

Now we show that $((h_1, h_2), c)$ can not block μ for some $h_1, h_2 \in H$ and $c = \{f, m\} \in C$. Assume for contradiction that μ is blocked by $((h_1, h_2), c)$.

First we show that $\mu(f) \underset{\sim}{R}_f h_1$. Suppose not. Since f proposes according to $\underset{\sim}{P}_f$ in Algorithm 4, it must be that f had proposed to h_1 and got rejected at some stage of the algorithm. This implies $d P^0_{h_1} f$ for all $d \in \mu(h_1)$, and hence, $((h_1, h_2), c)$ can not block μ, a contradiction.

Since $\mu(f) \underset{\sim}{R}_f h_1$ and couples' preferences satisfy responsiveness with respect to f, we have $(\mu(f), h_2) \underset{\sim}{R}_c (h_1, h_2)$. However, since $((h_1, h_2), c)$ blocks μ, we have $(h_1, h_2) \underset{\sim}{P}_c (\mu(f), \mu(m))$. Because $\mu(f) \underset{\sim}{R}_f h_1$, this implies that $(\mu(f), h_2) \underset{\sim}{P}_c (\mu(f), \mu(m))$, and hence, $h_2 \underset{\sim}{P}_{m|\mu(f)} \mu(m)$. Therefore, it follows that m had proposed to h_2 at an earlier stage of the algorithm and got rejected implying that $d P^0_{h_2} m$ for all $d \in \mu(h_2)$. However, this contradicts that $((h_1, h_2), c)$ blocks μ.

Since μ can not be blocked by single doctors or couples, μ is stable. ■

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