On percolation in a generalized backbend process*

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Abstract

We have generalized the idea of backbend in a nearest-neighbor oriented bond percolation process by considering a backbend sequence $\beta : \mathbb{Z}_+ \to \mathbb{Z}_+ \cup \{\infty\}$, and defining a β-backbend path from the origin as a path that never retreats further than $\beta(h)$ levels back from its record level h. We study the relationship between the critical probabilities of different percolation processes based on different backbend sequences on half-space, full-space, and half-slabs of the d-dimensional ($d \geq 2$) body-centered cubic (BCC) lattice. We also give sufficient conditions on the backbend sequences such that there will be no percolation at the critical probabilities of the corresponding percolation processes on half-space and full-space of the BCC lattice.

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1 Introduction

In this paper, we discuss an extension of the concept of nearest-neighbor oriented bond percolation with backbend on certain sublattices of the d-dimensional ($d \geq 2$) body-centered cubic (BCC) lattice. Roy et al. (1998) studied nearest-neighbor oriented bond percolation with backbend in the d-dimensional simple cubic lattice, where they worked with *b-backbend paths* for some $b \in \mathbb{Z}_+$.[1] A b-backbend path never retreats further than b levels back from its record level. We have generalized their idea of backbend as follows: instead of taking a constant value b for permissible backbend of a path, we consider a *backbend sequence* $\beta : \mathbb{Z}_+ \to \mathbb{Z}_+ \cup \{\infty\}$, and define a β-backbend path from the origin as a path that never retreats further than $\beta(h)$ levels back from its record level h. The generality of our framework allows us to embed important special cases; a β-backbend path can be an oriented path, a b-backbend path, or an ordinary (unoriented) path depending on the backbend sequence β.

We study the relationship between the critical probabilities of different percolation processes based on different backbend sequences on half-space, full-space, and half-slabs of the BCC lattice. We also give sufficient conditions on the backbend sequences such that there will be no percolation at the critical probabilities of the corresponding percolation processes on half-space and full-space of the BCC lattice.

2 Model

We denote the d-dimensional ($d \geq 2$) BCC lattice by $(\mathbb{V}, \mathbb{E})$, where the vertex set $\mathbb{V} := \{x \in \mathbb{Z}^d : x_i \equiv x_j \pmod 2 \text{ for all } i, j \in \{1, \dots, d\}\}$ and the edge set $\mathbb{E}$ contains all ***unordered*** pairs $\langle x, y\rangle$ of vertices $x, y \in \mathbb{V}$ with $|x_1 - y_1| = \cdots = |x_d - y_d| = 1$.[2] Each edge is independently ***open*** with probability $p \in [0, 1]$ and ***closed*** with probability $1 - p$. Let $\mathbb{P}_p$ denote the corresponding probability measure for the total configurations of all the edges.[3]

[1] We denote by $\mathbb{Z}_+$ the set $\mathbb{N} \cup \{0\}$.

[2] If we rotate the square lattice using the following rotational matrix

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

we will get the 2-dimensional BCC lattice. Thus, 2-dimensional BCC lattice is equivalent to the square lattice (see Grimmett (2013) for the formal definition of the square lattice).

[3] More formally, we consider the following probability space. As *sample space* we take $\Omega = \prod_{e \in \mathbb{E}} \{0, 1\}$, points of which are represented as $\omega = \{\omega(e) : e \in \mathbb{E}\}$ and called *configurations*; the value $w(e) = 0$ corresponds to e being closed, and $w(e) = 1$ corresponds to e being open. We take $\mathcal{F}$ to be the σ-algebra of subsets of Ω generated by the finite-dimensional cylinders. Finally, we take product measure with density p on $(\Omega, \mathcal{F})$; this is the measure

$$\mathbb{P}_p = \prod_{e \in \mathbb{E}} \mu_e$$

For $\widehat{\mathbb{V}} \subseteq \mathbb{V}$, a ***path in*** $\widehat{\mathbb{V}}$ is a sequence $x^0, x^1, \ldots, x^n$ of *distinct* vertices such that $x^i \in \widehat{\mathbb{V}}$ and $\langle x^i, x^{i+1} \rangle \in \mathbb{E}$. Note that paths are self-avoiding by definition. We call a path ***open*** if all of its edges are open. For a path $\pi = (x^0, \ldots, x^n)$, the ***record level*** attained by the path π till x^i, denoted by $h^i(\pi)$, is defined as $\max_{j \le i} x_d^j$. In words, the record level attained by the path π till x^i is the maximum value of the d-th coordinates of vertices till x^i.

A ***backbend sequence*** is a mapping $\beta : \mathbb{Z}_+ \to \mathbb{Z}_+ \cup \{\infty\}$. For ease of presentation, we write $\beta(i)$ as β_i. We introduce the notion of a β-backbend path.

Definition 2.1. For $\widehat{\mathbb{V}} \subseteq \mathbb{V}$, $x \in \widehat{\mathbb{V}}$ with $x_d \ge 0$, and a backbend sequence β, we say a path $\pi = (x^0 = x, \ldots, x^n)$ in $\widehat{\mathbb{V}}$ is a ***β-backbend path in $\widehat{\mathbb{V}}$ from x to x^n*** if for every i,

$$x_d^i \ge h^i(\pi) - \beta_{h^i(\pi)}.$$

In words, a β-backbend path never retreats further than β_h levels back from its record level h. Thus, if $\beta_i = 0$ for all $i \in \mathbb{Z}_+$, then the β-backbend path is an oriented path in the direction of the positive d-th coordinate axis (that is, in the direction of $(0, \ldots, 0, 1)$). If $\beta_i = b$ for all $i \in \mathbb{Z}_+$ and for some $b \in \mathbb{N}$, then the β-backbend path is a b-backbend path in the direction of the positive d-th coordinate axis.[4] Finally, if $\beta_i = \infty$ for all $i \in \mathbb{Z}_+$, then the β-backbend path is an ordinary (unoriented) path.

A backbend sequence β is ***k-cyclic*** for some $k \in \mathbb{N}$, if for all $i \in \{0, \ldots, k-1\}$, we have $\beta_{kn+i} = \beta_i \in \mathbb{Z}_+$ for all $n \in \mathbb{N}$ (that is, $\beta_i = \beta_{k+i} = \beta_{2k+i} = \cdots$ and $\beta_i \in \mathbb{Z}_+$). Note that a 1-cyclic backbend sequence is a constant sequence in $\mathbb{Z}_+$. Let $\tilde{\beta}$ be a backbend sequence. We say that $\tilde{\beta}$ ***converges to a k-cyclic backbend sequence*** β for some $k \in \mathbb{N}$, if for all $i \in \{0, \ldots, k-1\}$ we have $\lim_{n\to\infty} \tilde{\beta}_{kn+i} = \beta_i$, and say that $\tilde{\beta}$ ***converges from below to a k-cyclic backbend sequence*** β for some $k \in \mathbb{N}$, if for all $i \in \{0, \ldots, k-1\}$ we have $\lim_{n\to\infty} \tilde{\beta}_{kn+i} = \beta_i$ and $\tilde{\beta}_{kn+i} \le \beta_i$ for all $n \in \mathbb{Z}_+$.[5] For ease of reference, we call a backbend sequence *k-cyclic in the limit* if it converges to a k-cyclic backbend sequence, and *k-cyclic in the limit from below* if it converges from below to a k-cyclic backbend sequence. Note that a backbend sequence is 1-cyclic in the limit if and only if it converges to a finite integer.

Throughout this paper, we refer to the vertex $(0, \ldots, 0)$ as the *origin*. For $\widehat{\mathbb{V}} \subseteq \mathbb{V}$ and a backbend

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where μ_e is Bernoulli measure on $\{0, 1\}$, given by $\mu_e(\omega(e) = 0) = 1 - p$ and $\mu_e(\omega(e) = 1) = p$.

[4] See Roy et al. (1998) for the definition of a b-backbend path in the context of d-dimensional simple cubic lattice. Note that in their paper, the orientation is in the direction of $(1, \ldots, 1)$.

[5] Note that $\tilde{\beta}$ converges from below to a k-cyclic backbend sequence does *not* mean that for any $i \in \{0, \ldots, k-1\}$, the subsequence $(\tilde{\beta}_{kn+i})_{n \in \mathbb{Z}_+}$ is monotonically increasing.

sequence β, define the following random set of vertices:

$$C^{\beta}_{\widehat{\mathbb{V}}} := \{x \in \widehat{\mathbb{V}} : \text{there is an open } \beta\text{-backbend path in } \widehat{\mathbb{V}} \text{ from the origin to } x\}.$$

For $p \in [0,1]$, $\widehat{\mathbb{V}} \subseteq \mathbb{V}$, and a backbend sequence β, the ***percolation probability*** is defined as

$$\theta^{\beta}_{\widehat{\mathbb{V}}}(p) := \mathbb{P}_p\big(|C^{\beta}_{\widehat{\mathbb{V}}}| = \infty\big).$$

For $\widehat{\mathbb{V}} \subseteq \mathbb{V}$ and a backbend sequence β, the ***critical probability*** is defined as

$$p^{\beta}_c(\widehat{\mathbb{V}}) := \sup \{p : \theta^{\beta}_{\widehat{\mathbb{V}}}(p) = 0\}.$$

As we have explained earlier, by the definition of a β-backbend path, the critical probability $p^{\beta}_c(\widehat{\mathbb{V}})$ becomes the same as the critical probability of the (i) oriented percolation process in the direction of the positive d-th coordinate axis when $\beta_i = 0$ for all $i \in \mathbb{Z}_+$, (ii) b-backbend percolation process in the direction of the positive d-th coordinate axis when $\beta_i = b$ for all $i \in \mathbb{Z}_+$ and for some $b \in \mathbb{N}$, and (iii) ordinary (unoriented) percolation process when $\beta_i = \infty$ for all $i \in \mathbb{Z}_+$.

For $\widehat{\mathbb{V}} \subseteq \mathbb{V}$, we denote the critical probability of the oriented percolation process on $\widehat{\mathbb{V}}$ by $p^0_c(\widehat{\mathbb{V}})$, and the critical probability of the ordinary percolation process on $\widehat{\mathbb{V}}$ by $p_c(\widehat{\mathbb{V}})$.

3 Results

Let $\mathbb{H} := \{x \in \mathbb{V} : x_d \geq 0\}$ denote the *half-space*. For all $l \in \mathbb{N}$ and all $e \in \{2, \ldots, d\}$, let $\mathbb{Q}^e_l := \{x \in \mathbb{V} : x \in [-l,l]^{d-e} \times \mathbb{Z}^{e-1} \times \mathbb{Z}_+\}$ denote an *e-dimensional half-slab*.[6] For all $t \in \mathbb{N}$, let $\mathbb{S}_t := \{x \in \mathbb{V} : 0 \leq x_d \leq t\}$ denote a $d-1$*-dimensional slab*. Furthermore, for all $l_1, \ldots, l_d, r_1, \ldots, r_d \in \mathbb{Z} \cup \{-\infty, \infty\}$ with $l_i \leq r_i$ for all $i = 1, \ldots, d$, let us define

$$B\Big(\prod_{i=1}^{d}[l_i, r_i]\Big) := \Big\{x \in \mathbb{V} : x \in \prod_{i=1}^{d}[l_i, r_i]\Big\}.$$

For ease of presentation, we denote $B\Big(\prod\limits_{i=1}^{d-1}[l_i, r_i] \times [l,l]\Big)$ by $B\Big(\prod\limits_{i=1}^{d-1}[l_i, r_i] \times l\Big)$.

Furthermore, following our notational terminology, for $\widehat{\mathbb{V}} \subseteq \mathbb{V}$, $\mathbb{V}' \subseteq \mathbb{H}$, and a backbend sequence

[6] Note that for all $l \in \mathbb{N}$, $\mathbb{Q}^d_l = \mathbb{H}$.

β, define the random set of vertices

$$C^{\beta}_{\widehat{\mathbb{V}}}(\mathbb{V}') := \Big\{ y \in \widehat{\mathbb{V}} : \text{ there is an open } \beta\text{-backbend path in } \widehat{\mathbb{V}} \text{ from some } x \in \mathbb{V}' \text{ to } y \Big\}.$$

In what follows, we present a technical result which we will use in proving our main results of the paper.

Proposition 3.1. *Let β be a $\hat{k}$-cyclic backbend sequence for some $\hat{k} \in \mathbb{N}$. Suppose $0 < p < 1$ and $\theta^{\beta}_{\mathbb{H}}(p) > 0$. Then, there exist $l, r \in \mathbb{N}$ and $0 < \delta < p$ such that*

$$\mathbb{P}_{p-\delta}\left(\left| C^{\beta}_{\mathbb{Q}^2_l}\left(B\left([-r, r]^{d-1} \times 0 \right) \right) \right| = \infty \right) > 0.$$

The proof of this proposition is relegated to Appendix A.

3.1 Results on the half-space

Our next theorem establishes a relation between percolation probabilities of a $\tilde{\beta}$-backbend percolation process and a β-backbend percolation process on the half-space, where β is k-cyclic and $\tilde{\beta}$ converges from below to β.

Theorem 3.1. *Suppose a backbend sequence $\tilde{\beta}$ converges from below to a k-cyclic backbend sequence β for some $k \in \mathbb{N}$. Then, for all $p \in [0,1]$, we have $\theta^{\tilde{\beta}}_{\mathbb{H}}(p) > 0$ if and only if $\theta^{\beta}_{\mathbb{H}}(p) > 0$.*

Proof of Theorem 3.1. ***(If part)*** Suppose $\theta^{\beta}_{\mathbb{H}}(p) > 0$. It must be that $p > 0$. Since $\lim_{n\to\infty} \tilde{\beta}_{kn+i} = \beta_i$ for all $i \in \{0, \ldots, k-1\}$, there exists $n^* \in \mathbb{N}$ such that $\tilde{\beta}_{kn+i} = \beta_i$ for all $n \geq n^*$ and all $i \in \{0, \ldots, k-1\}$. Because β is k-cyclic, by the construction of the β-backbend percolation process, we have for all $y \in \mathbb{H}$ with $y_d = kn^*$,

$$\mathbb{P}_p\left(\left| C^{\beta}_{(\mathbb{H}+y)}(\{y\}) \right| = \infty \right) = \theta^{\beta}_{\mathbb{H}}(p).^{7} \tag{1}$$

Choose $y^* \in \mathbb{H}$ with $y^*_d = kn^*$ such that there exists an oriented path π from the origin to y^*.[8] As $p > 0$, the probability of π being open is positive.[9]

Since $\tilde{\beta}_n = \beta_n$ for all $n \geq kn^*$, $y^*_d = kn^*$, and π is an oriented path from the origin to y^*, it follows that π concatenated with any infinite β-backbend path in $\mathbb{H} + y^*$ from y^* produces an infinite

[7]For $x \in \mathbb{V}$ and $\widehat{\mathbb{V}} \subseteq \mathbb{V}$, we denote by $\widehat{\mathbb{V}} + x$ the set $\{y + x : y \in \widehat{\mathbb{V}}\}$.

[8]For instance, y^* can be taken as the vertex that is connected to the origin through the following oriented path: $((0,\ldots,0), (1,\ldots,1), (0,\ldots,0,2), (1,\ldots,1,3), \ldots, (0,\ldots,0,kn^*))$ if kn^* is even, and $((0,\ldots,0), (1,\ldots,1), (0,\ldots,0,2), \ldots, (1,\ldots,1,kn^*))$ if kn^* is odd.

[9]Note that since the path π is oriented, it will have kn^* edges, and hence $\mathbb{P}_p(\pi \text{ is open}) = p^{kn^*}$.

$\tilde{\beta}$-backbend path in $\mathbb{H}$ from the origin. Furthermore, by the construction of π, the event of π being open is independent of the event $\left\{\omega \in \Omega : \left|C^{\beta}_{(\mathbb{H}+y^*)}(\{y^*\})\right| = \infty\right\}$. Combining these two facts with (1), we have $\theta^{\tilde{\beta}}_{\mathbb{H}}(p) \geq \mathbb{P}_p(\pi \text{ is open}) \times \theta^{\beta}_{\mathbb{H}}(p)$. Because $\mathbb{P}_p(\pi \text{ is open}) > 0$ and $\theta^{\beta}_{\mathbb{H}}(p) > 0$, this implies $\theta^{\tilde{\beta}}_{\mathbb{H}}(p) > 0$. This completes the proof of the "if" part of Theorem 3.1.

(Only-if part) Suppose $\theta^{\tilde{\beta}}_{\mathbb{H}}(p) > 0$. By the assumptions on $\tilde{\beta}$ and β, for all $i \in \{0, \ldots, k-1\}$, we have $\tilde{\beta}_{kn+i} \leq \beta_i$ for all $n \in \mathbb{Z}_+$. This implies that every $\tilde{\beta}$-backbend path in $\mathbb{H}$ from the origin is a β-backbend path in $\mathbb{H}$ from the origin. Therefore, $\theta^{\beta}_{\mathbb{H}}(p) \geq \theta^{\tilde{\beta}}_{\mathbb{H}}(p)$. Since $\theta^{\tilde{\beta}}_{\mathbb{H}}(p) > 0$, this implies $\theta^{\beta}_{\mathbb{H}}(p) > 0$. This completes the proof of the "only-if" part of Theorem 3.1. ■

We obtain the following corollary (Corollary 3.1) from Theorem 3.1. It says that if a backbend sequence $\tilde{\beta}$ converges from below to a k-cyclic backbend sequence β, then on the half-space, the critical probability of $\tilde{\beta}$-backbend percolation process will be equal to that of β-backbend percolation process.

Corollary 3.1. *Suppose a backbend sequence $\tilde{\beta}$ converges from below to a k-cyclic backbend sequence β for some $k \in \mathbb{N}$. Then, $p^{\tilde{\beta}}_c(\mathbb{H}) = p^{\beta}_c(\mathbb{H})$.*

Note 3.1. It is worth mentioning that Corollary 3.1 does *not* hold if we relax its assumption by requiring that $\tilde{\beta}$ converges (not necessarily from below) to a k-cyclic backbend sequence β (see Example 3.1 for details). However, for this case, using the same argument as for the "if" part of Theorem 3.1, we have $p^{\tilde{\beta}}_c(\mathbb{H}) \leq p^{\beta}_c(\mathbb{H})$.

Example 3.1. Suppose $d = 3$. Let β be a 1-cyclic backbend sequence such that $\beta_i = 0$ for all $i \in \mathbb{Z}_+$. Suppose $\tilde{\beta}$ is a backbend sequence such that $\tilde{\beta}_0 = 0, \tilde{\beta}_1 = 1, \tilde{\beta}_2 = 2, \tilde{\beta}_3 = 3$, and $\tilde{\beta}_i = 0$ for all $i \geq 4$. Clearly, $\tilde{\beta}$ converges to β. By construction, $p^{\tilde{\beta}}_c(\mathbb{H}) \leq p_c(\mathbb{S}_3)$. Since $p_c(\mathbb{S}_3) = 0.21113018(38)$ (see Gliozzi et al. (2005) for details) and $p^{\beta}_c(\mathbb{H}) = 0.2873383(1)$ (see Perlsman and Havlin (2002) for details), the fact $p^{\tilde{\beta}}_c(\mathbb{H}) \leq p_c(\mathbb{S}_3)$ implies that $p^{\tilde{\beta}}_c(\mathbb{H}) < p^{\beta}_c(\mathbb{H})$.[10]

Remark 3.1. In two dimensions, Corollary 3.1 holds for the case where $\tilde{\beta}$ converges (not necessarily from below) to a k-cyclic backbend sequence β satisfying an additional property. See Proposition 3.3 for details.

The next theorem is our main result on half-space which says that if a backbend sequence is k-cyclic in the limit from below, then there will be no percolation on the half-space at the critical probability.

[10] Error bars in the last digit or digits are shown by numbers in parentheses. Thus, 0.21113018(38) signifies $0.21113018 \pm 0.00000038$.

Theorem 3.2. *Suppose a backbend sequence $\tilde{\beta}$ is k-cyclic in the limit from below for some $k \in \mathbb{N}$. Then, $\theta_{\mathbb{H}}^{\tilde{\beta}}(p_c^{\tilde{\beta}}(\mathbb{H})) = 0$.*

Before we start proving Theorem 3.2, we make a remark on the critical probability of any backbend percolation process on the half-space.

Remark 3.2. For every backbend sequence β, we have $p_c^{\beta}(\mathbb{H}) < 1$. To see this fix a backbend sequence β. By the definition of a β-backbend path, it follows that $p_c^{\beta}(\mathbb{H}) \le p_c^{0}(\mathbb{H})$. Using a similar argument as in Grimmett (2013), it can be verified that $p_c^{0}(\mathbb{H})$ decreases as d (the number of dimensions) increases. Furthermore, since 2-dimensional BCC lattice is equivalent to the square lattice, it is known that in two dimensions, $p_c^{0}(\mathbb{H}) < 1$ (see Durrett (1984), Durrett (1988), Balister et al. (1993)). Combining all these facts, we have $p_c^{\beta}(\mathbb{H}) < 1$.

Proof of Theorem 3.2. Let β be the k-cyclic backbend sequence such that $\tilde{\beta}$ converges from below to β. In view of Theorem 3.1, it is enough to show that $\theta_{\mathbb{H}}^{\beta}(p_c^{\beta}(\mathbb{H})) = 0$.

Assume for contradiction that $\theta_{\mathbb{H}}^{\beta}(p_c^{\beta}(\mathbb{H})) > 0$. Since $\theta_{\mathbb{H}}^{\beta}(p_c^{\beta}(\mathbb{H})) > 0$, we have $p_c^{\beta}(\mathbb{H}) > 0$. This, together with Remark 3.2, implies $0 < p_c^{\beta}(\mathbb{H}) < 1$. Moreover, since $0 < p_c^{\beta}(\mathbb{H}) < 1$ and $\theta_{\mathbb{H}}^{\beta}(p_c^{\beta}(\mathbb{H})) > 0$, by Proposition 3.1, there exist $l, r \in \mathbb{N}$ and $0 < \delta < p_c^{\beta}(\mathbb{H})$ such that

$$\mathbb{P}_{p_c^{\beta}(\mathbb{H})-\delta}\left(\left|C_{\mathbb{Q}_l^2}^{\beta}\left(B\left([-r,r]^{d-1} \times 0\right)\right)\right| = \infty\right) > 0. \tag{2}$$

Since $\mathbb{Q}_l^2 \subseteq \mathbb{H}$, (2) implies

$$\mathbb{P}_{p_c^{\beta}(\mathbb{H})-\delta}\left(\left|C_{\mathbb{H}}^{\beta}\left(B\left([-r,r]^{d-1} \times 0\right)\right)\right| = \infty\right) > 0. \tag{3}$$

Because $B\left([-r,r]^{d-1} \times 0\right)$ is a finite set, (3) implies that there exists some $x \in B\left([-r,r]^{d-1} \times 0\right)$ such that $\mathbb{P}_{p_c^{\beta}(\mathbb{H})-\delta}\left(|C_{\mathbb{H}}^{\beta}(\{x\})| = \infty\right) > 0$. Furthermore, by the construction of the β-backbend percolation process, we have $\mathbb{P}_{p_c^{\beta}(\mathbb{H})-\delta}\left(|C_{\mathbb{H}}^{\beta}| = \infty\right) = \mathbb{P}_{p_c^{\beta}(\mathbb{H})-\delta}\left(|C_{\mathbb{H}}^{\beta}(\{x\})| = \infty\right)$, and hence

$$\mathbb{P}_{p_c^{\beta}(\mathbb{H})-\delta}\left(|C_{\mathbb{H}}^{\beta}| = \infty\right) > 0. \tag{4}$$

However, since $\delta > 0$, (4) contradicts the definition of $p_c^{\beta}(\mathbb{H})$. This completes the proof of Theorem 3.2. ■

3.2 Results on the full-space

In this subsection, we consider the full-space and a particular class of backbend sequences. These backbend sequences have the property that they converge from below to a k-cyclic backbend sequence β satisfying the property that $\beta_1 \leq \beta_0 + 1, \ldots, \beta_{k-1} \leq \beta_{k-2} + 1$, and $\beta_0 \leq \beta_{k-1} + 1$ (that is, $\beta_{i+1} \leq \beta_i + 1$ for all $i \in \{0, \ldots, k-1\}$). For such backbend sequences, the first part of Theorem 3.3 says that the critical probability on the full-space will be the same as that on the half-space, and the second part of the theorem says that there will be no percolation on the full-space at the critical probability.

Theorem 3.3. *Let $k \in \mathbb{N}$ and suppose that a backbend sequence $\tilde{\beta}$ converges from below to a k-cyclic backbend sequence β satisfying $\beta_{i+1} \leq \beta_i + 1$ for all $i \in \{0, \ldots, k-1\}$. Then,*

(i) $p_c^{\tilde{\beta}}(\mathbb{V}) = p_c^{\tilde{\beta}}(\mathbb{H})$, *and*

(ii) $\theta_{\mathbb{V}}^{\tilde{\beta}}(p_c^{\tilde{\beta}}(\mathbb{V})) = 0$.

Proof of Theorem 3.3. We first prove a claim which we will use in the proof of the theorem.

Claim 3.1. *For all $p \in [0,1]$, $\theta_{\mathbb{V}}^{\beta}(p) > 0$ implies $\theta_{\mathbb{H}}^{\beta}(p) > 0$.*

Proof of Claim 3.1. Suppose $\theta_{\mathbb{V}}^{\beta}(p) > 0$. It must be that $p > 0$. Let $n^* \in k\mathbb{N}$ be such that $n^* \geq \max\{\beta_0, \ldots, \beta_{k-1}\}$.[11] Thus, n^* is such that any β-backbend path starting from a vertex y with $y_d = n^*$ will always be in the half-space $\mathbb{H}$. Because β is k-cyclic, by the construction of the β-backbend percolation process, we have for all $y \in \mathbb{H}$ with $y_d = n^*$,

$$\mathbb{P}_p\Big(\big|C_{\mathbb{V}}^{\beta}(\{y\})\big| = \infty\Big) = \theta_{\mathbb{V}}^{\beta}(p). \tag{5}$$

Choose $y^* \in \mathbb{H}$ with $y_d^* = n^*$ such that there exists an oriented path π from the origin to y^*. As $p > 0$, the probability of π being open is positive.

For each infinite β-backbend path $\hat{\pi}$ in $\mathbb{V}$ from y^*, we construct the path $\hat{\pi}^*$ by distinguishing the following cases.

(i) Suppose π and $\hat{\pi}$ have no common vertex other than y^*. Then, $\hat{\pi}^*$ is obtained by concatenating the paths π and $\hat{\pi}$.

(ii) Suppose π and $\hat{\pi}$ have common vertices other than y^*. Let z^* be the first vertex of π such that $z^* \in \hat{\pi}$. Then $\hat{\pi}^*$ is obtained by concatenating π_s and $\hat{\pi}_s$, where π_s is the sub-path of π from the origin to z^* and $\hat{\pi}_s$ is the sub-path of $\hat{\pi}$ from z^*.

[11] For $k \in \mathbb{N}$, we denote by $k\mathbb{N}$ the set $\{kn : n \in \mathbb{N}\}$.

To see that $\hat{\pi}^*$ is indeed a path, observe that $\hat{\pi}^*$ is self-avoiding by construction. We claim that for each infinite β-backbend path $\hat{\pi}$ in $\mathbb{V}$ from y^*, the path $\hat{\pi}^*$ constructed as above is an infinite β-backbend path in $\mathbb{H}$ from the origin. Clearly, $\hat{\pi}^*$ is an infinite path from the origin. Moreover, as we have mentioned earlier, by the choice of n^*, the path $\hat{\pi}$ is in $\mathbb{H}$, and hence the path $\hat{\pi}^*$ is in $\mathbb{H}$. We proceed to show that $\hat{\pi}^*$ is a β-backbend path.

Assume for contradiction that $\hat{\pi}^*$ is not a β-backbend path. Since $\hat{\pi}^*$ is not a β-backbend path, there exists a vertex x^* in $\hat{\pi}^*$ such that $x_d^* < h^* - \beta_{h^*}$, where h^* is the record level attained by the path $\hat{\pi}^*$ till x^*. Since π is an oriented path, by the construction of $\hat{\pi}^*$, it follows that x^* must be in $\hat{\pi}$. Let $\hat{h}$ be the record level attained by the path $\hat{\pi}$ till x^*. Because $\hat{\pi}$ is a β-backbend path, it must be that $x_d^* \geq \hat{h} - \beta_{\hat{h}}$. This, together with the fact $x_d^* < h^* - \beta_{h^*}$, yields

$$h^* - \beta_{h^*} > \hat{h} - \beta_{\hat{h}}. \tag{6}$$

Since π is an oriented path to y^* and $\hat{\pi}$ is a path from y^*, by the construction of $\hat{\pi}^*$, we have $\hat{h} \geq h^*$. The assumptions on β imply that $\beta_l - \beta_m \leq l - m$ for all $l \geq m$. Since $\hat{h} \geq h^*$, this yields $\beta_{\hat{h}} - \beta_{h^*} \leq \hat{h} - h^*$, a contradiction to (6). So, it must be that $\hat{\pi}^*$ is a β-backbend path.

Since for each infinite β-backbend path $\hat{\pi}$ in $\mathbb{V}$ from y^*, $\hat{\pi}^*$ is an infinite β-backbend path in $\mathbb{H}$ from the origin, by the construction of $\hat{\pi}^*$, we have

$$\theta^{\beta}_{\mathbb{H}}(p) \geq \mathbb{P}_p\Big(\pi \text{ is open and } \big|C^{\beta}_{\mathbb{V}}(\{y^*\})\big| = \infty\Big),$$

and hence, by FKG inequality (see Grimmett (2013) for details),

$$\theta^{\beta}_{\mathbb{H}}(p) \geq \mathbb{P}_p\big(\pi \text{ is open}\big) \times \mathbb{P}_p\Big(\big|C^{\beta}_{\mathbb{V}}(\{y^*\})\big| = \infty\Big).$$

By (5), this gives $\theta^{\beta}_{\mathbb{H}}(p) \geq \mathbb{P}_p\big(\pi \text{ is open}\big) \times \theta^{\beta}_{\mathbb{V}}(p)$. Since $\mathbb{P}_p\big(\pi \text{ is open}\big) > 0$ and $\theta^{\beta}_{\mathbb{V}}(p) > 0$, we have $\theta^{\beta}_{\mathbb{H}}(p) > 0$. This completes the proof of Claim 3.1. $\square$

Now, we complete the proof of Theorem 3.3. In view of Theorem 3.2, it is enough to show that for all $p \in [0,1]$, we have $\theta^{\tilde{\beta}}_{\mathbb{V}}(p) > 0$ if and only if $\theta^{\tilde{\beta}}_{\mathbb{H}}(p) > 0$.

Since $\mathbb{H} \subset \mathbb{V}$, it follows that for all $p \in [0,1]$, $\theta^{\tilde{\beta}}_{\mathbb{H}}(p) > 0$ implies $\theta^{\tilde{\beta}}_{\mathbb{V}}(p) > 0$. We proceed to show that for all $p \in [0,1]$, $\theta^{\tilde{\beta}}_{\mathbb{V}}(p) > 0$ implies $\theta^{\tilde{\beta}}_{\mathbb{H}}(p) > 0$. Suppose $\theta^{\tilde{\beta}}_{\mathbb{V}}(p) > 0$ for some $p \in [0,1]$. By the assumptions on $\tilde{\beta}$ and β, it follows that every $\tilde{\beta}$-backbend path in $\mathbb{V}$ from the origin is a β-backbend path in $\mathbb{V}$ from the origin. Therefore, $\theta^{\tilde{\beta}}_{\mathbb{V}}(p) > 0$ implies $\theta^{\beta}_{\mathbb{V}}(p) > 0$. This, together with the assumptions on β and Claim 3.1, yields $\theta^{\beta}_{\mathbb{H}}(p) > 0$. Moreover, since $\tilde{\beta}$ converges from below to β and $\theta^{\beta}_{\mathbb{H}}(p) > 0$, by Theorem 3.1, we have $\theta^{\tilde{\beta}}_{\mathbb{H}}(p) > 0$. This completes the proof of Theorem 3.3. ■

3.3 Results on half-slabs

Throughout this subsection, we assume that $d \geq 3$. Our next theorem (Theorem 3.4) establishes a relation between percolation probabilities of a $\tilde{\beta}$-backbend percolation process and a β-backbend percolation process on every e-dimensional half-slab, where β is k-cyclic and $\tilde{\beta}$ converges from below to β. The proof of the "only-if" part of the theorem is similar to that of Theorem 3.1, and the proof of the "if" part of the theorem follows from that of Theorem 3.1 by additionally imposing the following requirements: (i) $n^* \in 2\mathbb{N}$, (ii) $y^* = (0, \ldots, 0, kn^*)$, and (iii) π is in $\mathbb{Q}^e_l$. It can be verified that these additional requirements do not affect the logic of the proof of the "if" part of Theorem 3.1.

Theorem 3.4. *Suppose a backbend sequence $\tilde{\beta}$ converges from below to a k-cyclic backbend sequence β for some $k \in \mathbb{N}$. Then, for all $l \in \mathbb{N}$, all $e \in \{2, \ldots, d-1\}$, and all $p \in [0,1]$, we have $\theta^{\tilde{\beta}}_{\mathbb{Q}^e_l}(p) > 0$ if and only if $\theta^{\beta}_{\mathbb{Q}^e_l}(p) > 0$.*

We obtain the following corollary (Corollary 3.2) from Theorem 3.4. It says that if a backbend sequence $\tilde{\beta}$ converges from below to a k-cyclic backbend sequence β, then on every e-dimensional half-slab, the critical probability of $\tilde{\beta}$-backbend percolation process will be equal to that of β-backbend percolation process.

Corollary 3.2. *Suppose a backbend sequence $\tilde{\beta}$ converges from below to a k-cyclic backbend sequence β for some $k \in \mathbb{N}$. Then, for all $l \in \mathbb{N}$ and all $e \in \{2, \ldots, d-1\}$, we have $p^{\tilde{\beta}}_c(\mathbb{Q}^e_l) = p^{\beta}_c(\mathbb{Q}^e_l)$.*

Our next theorem says that if a backbend sequence is k-cyclic in the limit from below, then for all $e \in \{2, \ldots, d-1\}$, the limit of the critical probability on the e-dimensional half-slab $\mathbb{Q}^e_l$ as l goes to infinity will be equal to the critical probability on the half-space.

Theorem 3.5. *Suppose a backbend sequence $\tilde{\beta}$ is k-cyclic in the limit from below for some $k \in \mathbb{N}$. Then, for all $e \in \{2, \ldots, d-1\}$, we have $\lim_{l\to\infty} p^{\tilde{\beta}}_c(\mathbb{Q}^e_l) = p^{\tilde{\beta}}_c(\mathbb{H})$.*

Proof of Theorem 3.5. Fix an arbitrary $e \in \{2, \ldots, d-1\}$. Let β be the k-cyclic backbend sequence such that $\tilde{\beta}$ converges from below to β. In view of Corollary 3.1 and Corollary 3.2, it is enough to show that $\lim_{l\to\infty} p^{\beta}_c(\mathbb{Q}^e_l) = p^{\beta}_c(\mathbb{H})$.

Since $\mathbb{Q}^e_l \subset \mathbb{Q}^e_{l+1} \subset \mathbb{H}$ for all $l \in \mathbb{N}$, we have $\lim_{l\to\infty} p^{\beta}_c(\mathbb{Q}^e_l) \geq p^{\beta}_c(\mathbb{H})$. Assume for contradiction that $\lim_{l\to\infty} p^{\beta}_c(\mathbb{Q}^e_l) > p^{\beta}_c(\mathbb{H})$. Let $p \in [0,1]$ be such that $p^{\beta}_c(\mathbb{H}) < p < \lim_{l\to\infty} p^{\beta}_c(\mathbb{Q}^e_l)$. It must be that $0 < p < 1$

and $\theta^{\beta}_{\mathbb{H}}(p) > 0$. Since $0 < p < 1$ and $\theta^{\beta}_{\mathbb{H}}(p) > 0$, by Proposition 3.1, there exist $l^*, r^* \in \mathbb{N}$ and $0 < \delta < p$ such that

$$\mathbb{P}_{p-\delta}\left(\left|C^{\beta}_{\mathbb{Q}^2_{l^*}}\left(B\left([-r^*, r^*]^{d-1} \times 0\right)\right)\right| = \infty\right) > 0,$$

and hence

$$\mathbb{P}_{p}\left(\left|C^{\beta}_{\mathbb{Q}^2_{l^*}}\left(B\left([-r^*, r^*]^{d-1} \times 0\right)\right)\right| = \infty\right) > 0. \tag{7}$$

Because $B\left([-r^*, r^*]^{d-1} \times 0\right)$ is a finite set, (7) implies that there exists some $x \in B\left([-r^*, r^*]^{d-1} \times 0\right)$ such that $\mathbb{P}_p\left(\left|C^{\beta}_{(x+\mathbb{Q}^2_{l^*+r^*})}(\{x\})\right| = \infty\right) > 0$. Furthermore, by the construction of the β-backbend percolation process, we have $\mathbb{P}_p\left(\left|C^{\beta}_{\mathbb{Q}^2_{l^*+r^*}}\right| = \infty\right) = \mathbb{P}_p\left(\left|C^{\beta}_{(x+\mathbb{Q}^2_{l^*+r^*})}(\{x\})\right| = \infty\right)$, which implies $\mathbb{P}_p\left(\left|C^{\beta}_{\mathbb{Q}^2_{l^*+r^*}}\right| = \infty\right) > 0$. Since $\mathbb{Q}^2_{l^*+r^*} \subseteq \mathbb{Q}^e_{l^*+r^*}$, this gives

$$\mathbb{P}_p\left(\left|C^{\beta}_{\mathbb{Q}^e_{l^*+r^*}}\right| = \infty\right) > 0. \tag{8}$$

However, since $p < \lim_{l\to\infty} p^{\beta}_c(\mathbb{Q}^e_l)$ and $p^{\beta}_c(\mathbb{Q}^e_l)$ decreases as l increases, it must be that $p < p^{\beta}_c(\mathbb{Q}^e_{l^*+r^*})$, a contradiction to (8). This completes the proof of Theorem 3.5. ■

Note 3.2. As we have noted (Note 3.1) for Corollary 3.1, Theorem 3.5 also does *not* hold in general. More precisely, as Example 3.2 shows, if we relax the assumption of Theorem 3.5 by requiring that $\tilde{\beta}$ converges (not necessarily from below) to a k-cyclic backbend sequence β, then Theorem 3.5 does not hold on two-dimensional half-slabs when β satisfies the condition introduced in Subsection 3.2 (that is, $\beta_{i+1} \leq \beta_i + 1$ for all $i \in \{0, \ldots, k-1\}$).

We use the following proposition in Example 3.2. It provides an upper bound and a lower bound of the critical probability of a $\tilde{\beta}$-backbend percolation process on a two-dimensional half-slab when $\tilde{\beta}$ converges to a k-cyclic backbend sequence β satisfying $\beta_{i+1} \leq \beta_i + 1$ for all $i \in \{0, \ldots, k-1\}$.

Proposition 3.2. *Let $k \in \mathbb{N}$ and suppose that a backbend sequence $\tilde{\beta}$ converges to a k-cyclic backbend sequence β satisfying $\beta_{i+1} \leq \beta_i + 1$ for all $i \in \{0, \ldots, k-1\}$. Then, for all $l \in \mathbb{N}$, we have $p^{\beta}_c(\mathbb{Q}^2_{2l}) \leq p^{\tilde{\beta}}_c(\mathbb{Q}^2_l) \leq p^{\beta}_c(\mathbb{Q}^2_l)$.*

The proof of this proposition is relegated to Appendix B.

We are now ready to present our counter example.

Example 3.2. Suppose $d = 3$. Consider the backbend sequences β and $\tilde{\beta}$ given in Example 3.1, where β is 1-cyclic and $\tilde{\beta}$ converges to β. We have already shown that $p^{\tilde{\beta}}_c(\mathbb{H}) < p^{\beta}_c(\mathbb{H})$ in Example 3.1. By the assumption on β and $\tilde{\beta}$, it follows from Proposition 3.2 that $\lim_{l\to\infty} p^{\tilde{\beta}}_c(\mathbb{Q}^2_l) = \lim_{l\to\infty} p^{\beta}_c(\mathbb{Q}^2_l)$. Furthermore, by the assumption on β, it follows from Theorem 3.5 that $\lim_{l\to\infty} p_c^{\beta}(\mathbb{Q}_l^2) = p_c^{\beta}(\mathbb{H})$. The facts $p_c^{\tilde{\beta}}(\mathbb{H}) < p_c^{\beta}(\mathbb{H})$, $\lim_{l\to\infty} p_c^{\tilde{\beta}}(\mathbb{Q}_l^2) = \lim_{l\to\infty} p_c^{\beta}(\mathbb{Q}_l^2)$, and $\lim_{l\to\infty} p_c^{\beta}(\mathbb{Q}_l^2) = p_c^{\beta}(\mathbb{H})$ together imply $\lim_{l\to\infty} p_c^{\tilde{\beta}}(\mathbb{Q}_l^2) > p_c^{\tilde{\beta}}(\mathbb{H})$.

In the following, we present a result on the half-space in two dimensions (as we have mentioned in Remark 3.1). The proof of this result follows by using similar logic as for the proof of Proposition 3.2.

Proposition 3.3. *Suppose $d = 2$. Let $k \in \mathbb{N}$ and suppose that a backbend sequence $\tilde{\beta}$ converges to a k-cyclic backbend sequence β satisfying $\beta_{i+1} \leq \beta_i + 1$ for all $i \in \{0, \dots, k-1\}$. Then, $p_c^{\tilde{\beta}}(\mathbb{H}) = p_c^{\beta}(\mathbb{H})$.*

Appendix A Proof of Proposition 3.1

We first introduce some notations and make a remark to facilitate the proof.

For all $l, t \in \mathbb{Z}_+ \cup \{\infty\}$, all $u \in \{-1,1\}^{d-1}$, and all $v \in \{-1,1\}^{d-2}$, let us define

$$\begin{aligned}
T(l,t) &:= \Big\{x \in B\big([-l,l]^{d-1} \times [0,t]\big) : x_d = t\Big\},\\
F(l,t) &:= \Big\{x \in B\big([-l,l]^{d-1} \times [0,t]\big) : |x_i| = l \text{ for some } i \in \{1,\dots,d-1\}\Big\},\\
F_{(d-1)^+}(l,t) &:= \Big\{x \in F(l,t) : x_{d-1} = l\Big\},\\
T^u(l,t) &:= \Big\{x \in T(l,t) : 0 \leq x_i u_i \leq l \text{ for all } i \in \{1,\dots,d-1\}\Big\}, \text{ and}\\
F^v_{(d-1)^+}(l,t) &:= \Big\{x \in F_{(d-1)^+}(l,t) : 0 \leq x_i v_i \leq l \text{ for all } i \in \{1,\dots,d-2\}\Big\}.
\end{aligned}$$

The following remark follows from basic probability.

Remark A.1. Let $\alpha > 0$ and let A_1, A_2, A_3 be events such that A_2 and A_3 are disjoint with $\mathbb{P}_p(A_2) > 0$ and $\mathbb{P}_p(A_3) > 0$.

(a) Suppose $\mathbb{P}_p(A_1 \mid A_2) < \alpha$ and $\mathbb{P}_p(A_1 \mid A_3) < \alpha$. Then, $\mathbb{P}_p\big(A_1 \mid (A_2 \cup A_3)\big) < \alpha$.

(b) Suppose $\mathbb{P}_p(A_1 \mid A_2) > \alpha$ and $\mathbb{P}_p(A_1 \mid A_3) > \alpha$. Then, $\mathbb{P}_p\big(A_1 \mid (A_2 \cup A_3)\big) > \alpha$.

(c) Suppose $\mathbb{P}_p(A_1 \mid A_2) = \alpha$ and $\mathbb{P}_p(A_1 \mid A_3) = \alpha$. Then, $\mathbb{P}_p\big(A_1 \mid (A_2 \cup A_3)\big) = \alpha$.

A.1 The proof

Let $\eta^* \in (0,1)$ be sufficiently small. Choose $i^* \in \mathbb{N}$ with $i^* > 10$, and $\epsilon^* \in (0,1)$ such that $(1-\epsilon^*)^{2i^*} > 1 - \eta^*$. We prove Proposition 3.1 by distinguishing the following two cases.

Case 1: Suppose $\theta^{\beta}_{\mathbb{S}_t}(p) = 0$ for all $t \in \mathbb{N}$.

Since $\theta^{\beta}_{\mathbb{H}}(p) > 0$, by a standard argument (see Liggett (2012), Theorem 1.10(d) in Chapter VI for details), there exists $r^* \in 2\hat{k}\mathbb{N}$ with $2r^* \geq \max\{\beta_0, \ldots, \beta_{\hat{k}-1}\}$ such that

$$\mathbb{P}_p\left(\left|C^{\beta}_{\mathbb{H}}\left(B\left([-r^*, r^*]^{d-1} \times 0\right)\right)\right| = \infty\right) > 1 - \frac{1}{2^{2^{d-1}}}\left(\frac{\epsilon^*}{5}\right)^{d2^{d-1}}. \tag{A.1}$$

For ease of presentation, let us denote $B\left([-r^*, r^*]^{d-1} \times 0\right)$ by D^*. We define the following terms involving ϵ^* and r^* for our next lemma.

- Let $\alpha^* := \min\left\{\begin{array}{l}\mathbb{P}_p\left(B\left([-r^*, r^*]^{d-1} \times [2r^*, 2r^* + 2\hat{k}]\right) \subseteq C^0_{B\left([-r^*,r^*]^{d-1}\times[0,2r^*+2\hat{k}]\right)}\right), \\ \mathbb{P}_p\left(B\left([0, 2r^*] \times [-r^*, r^*]^{d-2} \times [2r^*, 2r^* + 2\hat{k}]\right) \subseteq C^0_{B\left([0,2r^*]\times[-r^*,r^*]^{d-2}\times[0,2r^*+2\hat{k}]\right)}\right)\end{array}\right\}$.[12]
  Clearly, $\alpha^* > 0$.

- Let $m^* \in \mathbb{N}$ be such that $(1 - \alpha^*)^{m^*} < \frac{\epsilon^*}{5}$. The implication of m^* is that if one performs at least m^* independent trials with the probability of success in each trial being α^*, then the probability that there is at least one success exceeds $1 - \frac{\epsilon^*}{5}$.

- Let $n^* \in \mathbb{N}$ be such that $n^* \geq m^*(8r^* + 8\hat{k} + 1)^d$. The purpose of n^* is to ensure that in any subset of $\mathbb{V}$ having size n^* or larger, there are at least m^* vertices such the L^∞-distance between any two of them is at least $(4r^* + 4\hat{k} + 1)$.[13]

Lemma A.1. *There exist $l^*, t^* \in 2\hat{k}\mathbb{N}$ with $l^* \geq r^*$ such that*

(i) $\mathbb{P}_p\left(\left|C^{\beta}_{B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)}(D^*) \cap T^u(l^*, t^*)\right| \geq n^*\right) > 1 - \frac{\epsilon^*}{5}$ *for all $u \in \{-1, 1\}^{d-1}$, and*

(ii) $\mathbb{P}_p\left(\left|C^{\beta}_{B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)}(D^*) \cap F^v_{(d-1)^+}(l^*, t^*)\right| \geq n^*\right) > 1 - \frac{\epsilon^*}{5}$ *for all $v \in \{-1, 1\}^{d-2}$.*

The proof of this lemma is relegated to Appendix A.1.1.

Suppose l^* and t^* are such that the statement of Lemma A.1 holds. Let $s^* := t^* + 2r^* + 2\hat{k}$. Since $2r^* \geq \max\{\beta_0, \ldots, \beta_{\hat{k}-1}\}$ and $r^*, t^* \in 2\hat{k}\mathbb{N}$, by the construction of s^*, we have $s^* \in 2\hat{k}\mathbb{N}$ and $s^* > \max\{\beta_0, \ldots, \beta_{\hat{k}-1}\}$.

Lemma A.2. *There exists $0 < \delta < p$ such that for all $x \in B\left([-l^*, l^*]^{d-2} \times 0 \times 0\right)$, we have*

$$\mathbb{P}_{p-\delta}\left(\begin{array}{l}(D^* + z) \subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-l^*,3l^*]\times[0,z_d]\right)}(D^* + x) \\ \text{for some } z \in B\left([-l^*, l^*]^{d-2} \times [l^*, 2l^*] \times [s^*, 2s^*]\right) \text{ with } z_d \in 2\hat{k}\mathbb{N}\end{array}\right) > 1 - \epsilon^*.$$

[12] By $C^0_{\hat{\mathbb{V}}}$, we denote $C^{\beta}_{\hat{\mathbb{V}}}$ where $\beta_i = 0$ for all $i \in \mathbb{Z}_+$.
[13] $L^\infty(x, y) := \max_i \{|x_i - y_i|\}$.

The proof of this lemma is relegated to Appendix A.1.2.

Suppose δ^* is such that the statement of Lemma A.2 holds.

Lemma A.3. *For all* $x \in B\big([-l^*,l^*]^{d-2} \times [-2l^*,2l^*] \times [0,2s^*]\big)$ *with* $x_d \in 2\hat{k}\mathbb{Z}_+$, *we have*

$$\mathbb{P}_{p-\delta^*}\begin{pmatrix} (D^*+z) \subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-3l^*,4l^*]\times[x_d,z_d]\right)}(D^*+x) \\ \text{for some } z \in B\big([-l^*,l^*]^{d-2} \times [-l^*,3l^*] \times [2s^*,4s^*]\big) \text{ with } z_d \in 2\hat{k}\mathbb{N} \end{pmatrix} > (1-\epsilon^*)^2.$$

The proof of this lemma is relegated to Appendix A.1.3.

For all $i \in \mathbb{N}$, we denote the set of vertices $B\big([-l^*,l^*]^{d-2} \times [(i-2)l^*,(i+2)l^*] \times [2is^*,2(i+1)s^*]\big)$ by B_i^+. For all $i \in \mathbb{N}$ and all $z \in B_i^+$, let us define

$$\mathcal{R}^+(z) := \left\{ y \in \mathbb{H} : \begin{array}{l} -2l^* \leq y_j \leq 2l^* \text{ for all } j \in \{1,\ldots,d-2\}, \\ -5l^* + \dfrac{l^*}{2s^*}y_d \leq y_{d-1} \leq 5l^* + \dfrac{l^*}{2s^*}y_d, \text{ and } 0 \leq y_d \leq z_d \end{array} \right\}.$$

For all $i \in \mathbb{N}$ and all $x \in B\big([-l^*,l^*]^{d-2} \times [-2l^*,2l^*] \times [0,2s^*]\big)$ with $x_d \in 2\hat{k}\mathbb{Z}_+$, let us define the following event:

$$G_i^+(x) := \Big\{\omega \in \Omega : (D^*+z) \subseteq C^{\beta}_{\mathcal{R}^+(z)}(D^*+x) \text{ for some } z \in B_i^+ \text{ with } z_d \in 2\hat{k}\mathbb{N}\Big\}.$$

Lemma A.4. *For all* $i \in \mathbb{N}$ *and all* $x \in B\big([-l^*,l^*]^{d-2} \times [-2l^*,2l^*] \times [0,2s^*]\big)$ *with* $x_d \in 2\hat{k}\mathbb{Z}_+$, *we have* $\mathbb{P}_{p-\delta^*}\big(G_i^+(x)\big) > (1-\epsilon^*)^{2i}$.

The proof of this lemma is relegated to Appendix A.1.4.

For all $i \in \mathbb{N}$, we denote the set of vertices $B\big([-l^*,l^*]^{d-2} \times [(-i-2)l^*,(-i+2)l^*] \times [2is^*,2(i+1)s^*]\big)$ by B_i^-. For all $i \in \mathbb{N}$ and all $z \in B_i^-$, let us define

$$\mathcal{R}^-(z) := \left\{ y \in \mathbb{H} : \begin{array}{l} -2l^* \leq y_j \leq 2l^* \text{ for all } j \in \{1,\ldots,d-2\}, \\ -5l^* - \dfrac{l^*}{2s^*}y_d \leq y_{d-1} \leq 5l^* - \dfrac{l^*}{2s^*}y_d, \text{ and } 0 \leq y_d \leq z_d \end{array} \right\}.$$

For all $i \in \mathbb{N}$ and all $x \in B\big([-l^*,l^*]^{d-2} \times [-2l^*,2l^*] \times [0,2s^*]\big)$ with $x_d \in 2\hat{k}\mathbb{Z}_+$, let us define the following event:

$$G_i^-(x) := \Big\{\omega \in \Omega : (D^*+z) \subseteq C^{\beta}_{\mathcal{R}^-(z)}(D^*+x) \text{ for some } z \in B_i^- \text{ with } z_d \in 2\hat{k}\mathbb{N}\Big\}.$$

By the assumptions on i^* and ϵ^*, and the construction of the β-backbend percolation process, Lemma A.4 implies that for all $x \in B\big([-l^*, l^*]^{d-2} \times [-2l^*, 2l^*] \times [0, 2s^*]\big)$ with $x_d \in 2\hat{k}\mathbb{Z}_+$,

$$\mathbb{P}_{p-\delta^*}(G_{i^*}^+(x)) > 1 - \eta^* \text{ and } \mathbb{P}_{p-\delta^*}(G_{i^*}^-(x)) > 1 - \eta^*. \tag{A.2}$$

Lemma A.5. $\mathbb{P}_{p-\delta^*}\left(\left|C^{\beta}_{\mathbb{Q}^2_{2l^*}}(D^*)\right| = \infty\right) > 0.$

Since β is $\hat{k}$-cyclic, $s^* > \max\{\beta_0, \dots, \beta_{\hat{k}-1}\}$, $i^* > 10$, and η^* is sufficiently small, the proof of Lemma A.5 follows from (A.2) by using a similar logic as for the proof of Lemma 21 in Bezuidenhout et al. (1990).

Now, the proof of Proposition 3.1 for Case 1 follows from Lemma A.5.

Case 2: Suppose $\theta^{\beta}_{\mathbb{S}_t}(p) > 0$ for some $t \in \mathbb{N}$.
Fix $\hat{t} \in \mathbb{N}$ such that $\theta^{\beta}_{\mathbb{S}_{\hat{t}}}(p) > 0$. By a standard argument (see Liggett (2012), Theorem 1.10(d) in Chapter VI for details), this implies that there exists $r^* \in 2\hat{k}\mathbb{N}$ with $2r^* \geq \max\{\hat{t}, \beta_0, \dots, \beta_{\hat{k}-1}\}$ such that

$$\mathbb{P}_p\left(\left|C^{\beta}_{\mathbb{S}_{\hat{t}}}\left(B\big([-r^*, r^*]^{d-1} \times 0\big)\right)\right| = \infty\right) > 1 - \left(\frac{\epsilon^*}{2}\right)^{(d-1)2^{d-1}}. \tag{A.3}$$

For ease of presentation, we denote $B\big([-r^*, r^*]^{d-1} \times 0\big)$ by D^*. Let $t^* := 2r^*$. Since $r^* \in 2\hat{k}\mathbb{N}$ and $2r^* \geq \max\{\hat{t}, \beta_0, \dots, \beta_{\hat{k}-1}\}$, this implies $t^* \in 2\hat{k}\mathbb{N}$ and $t^* \geq \max\{\hat{t}, \beta_0, \dots, \beta_{\hat{k}-1}\}$. Furthermore, since $t^* \geq \hat{t}$, $\theta^{\beta}_{\mathbb{S}_{\hat{t}}}(p) > 0$ implies $\theta^{\beta}_{\mathbb{S}_{t^*}}(p) > 0$ and (A.3) implies

$$\mathbb{P}_p\left(\left|C^{\beta}_{\mathbb{S}_{t^*}}(D^*)\right| = \infty\right) > 1 - \left(\frac{\epsilon^*}{2}\right)^{(d-1)2^{d-1}}. \tag{A.4}$$

Let us define the following terms involving ϵ^* and r^* for our next lemma.

- Let $\alpha^* := \mathbb{P}_p\Big(B\big([0, 2r^*] \times [-r^*, r^*]^{d-2} \times [2r^* + 2\hat{k}, 2r^* + 4\hat{k}]\big) \subseteq C^0_{B\left([0,2r^*] \times [-r^*, r^*]^{d-2} \times [0, 2r^* + 4\hat{k}]\right)}\Big)$. Clearly, $\alpha^* > 0$.
- Let $m^* \in \mathbb{N}$ be such that $(1 - \alpha^*)^{m^*} < \frac{\epsilon^*}{2}$. The implication of m^* is that if one performs at least m^* independent trials with probability of success in each trial being α^*, the probability that there is at least one success (out of all trials) exceeds $1 - \frac{\epsilon^*}{2}$.
- Let $n^* \in \mathbb{N}$ be such that $n^* \geq m^*(8r^* + 16\hat{k} + 1)^d$. The purpose of n^* is to ensure that in any subset of $\mathbb{V}$ having size n^* or larger, there are at least m^* vertices such the L^{∞}-distance between any two of them is at least $(4r^* + 8\hat{k} + 1)$.

Lemma A.6. *There exists $l^* \in 2\hat{k}\mathbb{N}$ with $l^* \geq r^*$ such that for all $v \in \{-1,1\}^{d-2}$, we have*

$$\mathbb{P}_p\left(\left|C^{\beta}_{B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)}(D^*) \cap F^{v}_{(d-1)^+}(l^*,t^*)\right| \geq n^*\right) > 1 - \frac{\epsilon^*}{2}.$$

The proof of this lemma is relegated to Appendix A.1.5.

Suppose l^* is such that the statement of Lemma A.6 holds. Let $s^* := t^* + 2\hat{k}$. Since $t^* \in 2\hat{k}\mathbb{N}$ and $t^* \geq \max\{\beta_0, \ldots, \beta_{\hat{k}-1}\}$, by the construction of s^*, we have $s^* \in 2\hat{k}\mathbb{N}$ and $s^* > \max\{\beta_0, \ldots, \beta_{\hat{k}-1}\}$.

Lemma A.7. *There exists $0 < \delta < p$ such that for all $x \in B\left([-l^*,l^*]^{d-2} \times 0 \times 0\right)$, we have*

$$\mathbb{P}_{p-\delta}\begin{pmatrix} \left(D^* + z\right) \subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-l^*,3l^*]\times[0,z_d]\right)}(D^* + x) \\ \textit{for some } z \in B\left([-l^*,l^*]^{d-2} \times [l^*,2l^*] \times [s^*,2s^*]\right) \textit{ with } z_d \in 2\hat{k}\mathbb{N} \end{pmatrix} > 1 - \epsilon^*.$$

The proof of this lemma is relegated to Appendix A.1.6.

Suppose δ^* is such that the statement of Lemma A.7 holds. Then, the proof of Proposition 3.1 for Case 2 follows from Lemma A.7 by using an argument similar to the one which we have used to complete the proof of Proposition 3.1 for Case 1 from Lemma A.2.

A.1.1 Proof of Lemma A.1

Claim A.1. *Let $N \in \mathbb{N}$. Then,*

$$\mathbb{P}_p\left(\left|C^{\beta}_{\mathbb{H}}(D^*)\right| = \infty \textit{ and } \left|\left(C^{\beta}_{\mathbb{S}_t}(D^*) \cap T(\infty,t)\right)\right| < N \textit{ for infinitely many } t \in \mathbb{N}\right) = 0.$$

Proof of Claim A.1. Define the following events:

$$B_1 := \left\{\omega \in \Omega : \left|C^{\beta}_{\mathbb{H}}(D^*)\right| = \infty \text{ and } \left|\left(C^{\beta}_{\mathbb{S}_t}(D^*) \cap T(\infty,t)\right)\right| < N \text{ for infinitely many } t \in \mathbb{N}\right\},$$

$$B_2 := \left\{\omega \in \Omega : \begin{array}{l} \left|\left(C^{\beta}_{\mathbb{S}_{t'}}(D^*) \cap T(\infty,t')\right)\right| > 0 \;\forall\; t' \in \mathbb{N} \text{ and} \\ \left|\left(C^{\beta}_{\mathbb{S}_t}(D^*) \cap T(\infty,t)\right)\right| < N \text{ for infinitely many } t \in \mathbb{N} \end{array}\right\}, \text{ and}$$

$$B_3 := \left\{\omega \in \Omega : \left|C^{\beta}_{\mathbb{H}}(D^*)\right| = \infty \text{ and } \left|\left(C^{\beta}_{\mathbb{S}_{t'}}(D^*) \cap T(\infty,t')\right)\right| = 0 \text{ for some } t' \in \mathbb{N}\right\}.$$

By the constructions of B_1, B_2, and B_3, along with the construction of the β-backbend percolation process, we have $B_1 = B_2 \cup B_3$ and $B_2 \cap B_3 = \emptyset$.

First, we show that $\mathbb{P}_p(B_3) = 0$. For all $t \in \mathbb{N}$, define

$$B_3^t := \left\{ \omega \in \Omega : \left| C_{\mathbb{H}}^{\beta}(D^*) \right| = \infty \text{ and } \left| \left(C_{\mathbb{S}_t}^{\beta}(D^*) \cap T(\infty, t) \right) \right| = 0 \right\}.$$

By the assumption for Case 1, we have $\theta_{\mathbb{S}_t}^{\beta}(p) = 0$ for all $t \in \mathbb{N}$. Since D^* is a finite set of vertices, by the construction of the β-backbend percolation process, this implies $\mathbb{P}_p(B_3^t) = 0$ for all $t \in \mathbb{N}$. By construction, we have $B_3 = \bigcup_{t \in \mathbb{N}} B_3^t$. Since $\mathbb{P}_p(B_3^t) = 0$ for all $t \in \mathbb{N}$, this implies $\mathbb{P}_p(B_3) = 0$.

Next, we show that $\mathbb{P}_p(B_2) = 0$. For all $t \in \mathbb{Z}_+$, let $\mathcal{F}_t$ be the σ-algebra generated by $\left\{ \left| \left(C_{\mathbb{S}_{t'}}^{\beta}(D^*) \cap T(\infty, t') \right) \right| : 0 \leq t' \leq t \right\}$.[14] Fix $t \in \mathbb{N}$ and $F \in \mathcal{F}_t$. For a vertex $x \in T(\infty, t)$, define the following event:

$$A_x := \left\{ \omega \in \Omega : \langle x, y \rangle \text{ is closed whenever } y \in T(\infty, t+1) \right\}.$$

Clearly, $\mathbb{P}_p(A_x) = (1-p)^{2^{d-1}}$ for all $x \in T(\infty, t)$. Fix $S \subseteq T(\infty, t)$ such that $|S| < N$. Then,

$$\begin{aligned}
&\mathbb{P}_p\left(\left| \left(C_{\mathbb{S}_{t+1}}^{\beta}(D^*) \cap T(\infty, t+1) \right) \right| = 0 \;\middle|\; F \text{ and } \left(C_{\mathbb{S}_t}^{\beta}(D^*) \cap T(\infty, t) \right) = S \right) \\
&= \mathbb{P}_p\left(\bigcap_{x \in S} A_x \right) \qquad \text{(by the construction of the } \beta\text{-backbend percolation process)} \\
&= \prod_{x \in S} \mathbb{P}_p(A_x) \qquad \text{(since each edge is independent)} \\
&> (1-p)^{2^{d-1} N}.
\end{aligned}$$

This, along with Remark A.1, yields

$$\mathbb{P}_p\left(\left| \left(C_{\mathbb{S}_{t+1}}^{\beta}(D^*) \cap T(\infty, t+1) \right) \right| > 0 \;\middle|\; F \text{ and } \left| \left(C_{\mathbb{S}_t}^{\beta}(D^*) \cap T(\infty, t) \right) \right| < N \right) < 1 - (1-p)^{2^{d-1} N}. \tag{A.5}$$

Define the sequence of stopping times $(\tau_m)_{m \in \mathbb{N}}$ as follows. For all $\omega \in \Omega$,

(i) $\tau_1(\omega) := \min \left\{ t \in \mathbb{N} : \left| \left(C_{\mathbb{S}_t}^{\beta}(D^*) \cap T(\infty, t) \right) \right| < N \right\}$ and

(ii) for all $m \in \mathbb{N} \setminus \{1\}$, $\tau_m(\omega) := \min \left\{ t > \tau_{m-1}(\omega) : \left| \left(C_{\mathbb{S}_t}^{\beta}(D^*) \cap T(\infty, t) \right) \right| < N \right\}$,

where $\min \emptyset = \infty$. By the definition of $\{\tau_m\}_{m \in \mathbb{N}}$, along with (A.5) and Remark A.1, for all $m \in \mathbb{N}$ and

[14]Note that $\left| \left(C_{\mathbb{S}_t}^{\beta}(D^*) \cap T(\infty, t) \right) \right|$ is a random variable for all $t \in \mathbb{Z}_+$.

all $F \in \mathcal{F}_{\tau_m}$, we have

$$\mathbb{P}_p\left(\left|\left(C^{\beta}_{\mathbb{S}_{\tau_m+1}}(D^*) \cap T(\infty, \tau_m+1)\right)\right| > 0 \;\middle|\; F \text{ and } \tau_m < \infty\right) < 1-(1-p)^{2^{d-1}N}. \tag{A.6}$$

Now,

$$\begin{aligned}
\mathbb{P}_p(B_2) &= \mathbb{P}_p\left(\left|\left(C^{\beta}_{\mathbb{S}_{t'}}(D^*) \cap T(\infty, t')\right)\right| > 0 \;\forall\, t' \in \mathbb{N} \text{ and } \tau_m < \infty \;\forall\, m \in \mathbb{N}\right) \\
&= \mathbb{P}_p(\tau_1 < \infty) \times \mathbb{P}_p\left(\left|\left(C^{\beta}_{\mathbb{S}_{\tau_1+1}}(D^*) \cap T(\infty, \tau_1+1)\right)\right| > 0 \;\middle|\; \tau_1 < \infty\right) \\
&\quad \times \mathbb{P}_p\left(\tau_2 < \infty \;\middle|\; \tau_1 < \infty, \left|\left(C^{\beta}_{\mathbb{S}_{\tau_1+1}}(D^*) \cap T(\infty, \tau_1+1)\right)\right| > 0\right) \\
&\quad \times \mathbb{P}_p\left(\left|\left(C^{\beta}_{\mathbb{S}_{\tau_2+1}}(D^*) \cap T(\infty, \tau_2+1)\right)\right| > 0 \;\middle|\; \begin{array}{l} \tau_1 < \infty, \left|\left(C^{\beta}_{\mathbb{S}_{\tau_1+1}}(D^*) \cap T(\infty, \tau_1+1)\right)\right| > 0, \\ \tau_2 < \infty \end{array}\right) \\
&\quad \times \cdots .
\end{aligned} \tag{A.7}$$

Since $p < 1$, we have $1-(1-p)^{2^{d-1}N} < 1$. This, together with (A.6) and (A.7), implies $\mathbb{P}_p(B_2) = 0$.

Since $B_1 = B_2 \cup B_3$ and $B_2 \cap B_3 = \emptyset$, the facts $\mathbb{P}_p(B_2) = 0$ and $\mathbb{P}_p(B_3) = 0$ together imply $\mathbb{P}_p(B) = 0$. This completes the proof of Claim A.1. □

Let $k^* \in \mathbb{N}$ be such that $\left(1-(1-p^{2\hat{k}})^{k^*}\right)^{n^*} > 1-\frac{\epsilon^*}{10}$.[15] Define $A := \left\{\omega \in \Omega : \left|C^{\beta}_{\mathbb{H}}(D^*)\right| = \infty\right\}$ and $A_k := \left\{\omega \in \Omega : \left|\left(C^{\beta}_{\mathbb{S}_t}(D^*) \cap T(\infty, t)\right)\right| \geq 2^{d-1}k^*n^* \;\forall\, t \geq k\right\}$ for all $k \in \mathbb{N}$. By construction, we have

$$A_k \underset{k\to\infty}{\uparrow} \left(A \setminus \left\{\omega : \left|C^{\beta}_{\mathbb{H}}(D^*)\right| = \infty \text{ and } \left|\left(C^{\beta}_{\mathbb{S}_t}(D^*) \cap T(\infty, t)\right)\right| < 2^{d-1}k^*n^* \text{ for infinitely many } t \in \mathbb{N}\right\}\right),$$

which, together with Claim A.1, yields $\mathbb{P}_p(A_k) \underset{k\to\infty}{\uparrow} \mathbb{P}_p(A)$. This, along with (A.1), implies that there exists $t_1 \in 2\hat{k}\mathbb{N}$ such that $\mathbb{P}_p(A_{t_1}) > 1-\frac{1}{2^{2^{d-1}}}\left(\frac{\epsilon^*}{5}\right)^{d2^{d-1}}$. This, together with the construction of A_{t_1}, yields

$$\mathbb{P}_p\left(\left|\left(C^{\beta}_{\mathbb{S}_{t_1}}(D^*) \cap T(\infty, t_1)\right)\right| \geq 2^{d-1}k^*n^*\right) > 1-\frac{1}{2^{2^{d-1}}}\left(\frac{\epsilon^*}{5}\right)^{d2^{d-1}}. \tag{A.8}$$

Furthermore, by the assumption for Case 1, we have $\theta^{\beta}_{\mathbb{S}_{t_1}}(p) = 0$. Since D^* is a finite set of vertices, this

[15]Since p, n^*, and ϵ^* are already fixed, the facts $p > 0$ and $1-\left(1-\frac{\epsilon^*}{10}\right)^{\frac{1}{n^*}} > 0$ together imply k^* is well-defined.

implies

$$\mathbb{P}_p\left(\left|\left(C^{\beta}_{B\left([-l,l]^{d-1}\times[0,t_1]\right)}(D^*)\cap T(l,t_1)\right)\right|\geq 2^{d-1}k^*n^*\right)\underset{l\to\infty}{\uparrow}\mathbb{P}_p\left(\left|\left(C^{\beta}_{\mathbb{S}_{t_1}}(D^*)\cap T(\infty,t_1)\right)\right|\geq 2^{d-1}k^*n^*\right).$$

Since $\left(\frac{\epsilon^*}{10}\right)^{2^{d-1}} > \frac{1}{2^{2^{d-1}}}\left(\frac{\epsilon^*}{5}\right)^{d2^{d-1}}$, this, along with (A.8), implies that there exists $l(t_1)\in 2\hat{k}\mathbb{N}$ such that

$$\mathbb{P}_p\left(\left|\left(C^{\beta}_{B\left([-l(t_1),l(t_1)]^{d-1}\times[0,t_1]\right)}(D^*)\cap T(l(t_1),t_1)\right)\right|\geq 2^{d-1}k^*n^*\right)>1-\left(\frac{\epsilon^*}{10}\right)^{2^{d-1}}.$$

Define

$$s(l(t_1),t_1):=\min_{s\in 2\hat{k}\mathbb{N}}\left\{s\geq t_1:\mathbb{P}_p\left(\left|\left(C^{\beta}_{B\left([-l(t_1),l(t_1)]^{d-1}\times[0,s]\right)}(D^*)\cap T(l(t_1),s)\right)\right|\geq 2^{d-1}k^*n^*\right)\leq 1-\left(\frac{\epsilon^*}{10}\right)^{2^{d-1}}\right\},$$

where $\min\emptyset=\infty$.[16] Since $p<1$ and $B(l(t_1),\infty)$ is a one-dimensional cylinder, we have $s(l(t_1),t_1)\in 2\hat{k}\mathbb{N}$.

Construct three sequences of positive integers $(l_k)_{k\in\mathbb{N}}$, $(t_k)_{k\in\mathbb{N}}$, and $(s_k)_{k\in\mathbb{N}}$ as follows. Let $l_1=\max\left\{l(t_1),r^*\right\}$ and $s_1=s(l_1,t_1)$. Suppose $k\geq 1$ and suppose that $l_1,\ldots,l_k$, $t_1,\ldots,t_k$ and $s_1,\ldots,s_k$ are constructed. Choose $t_{k+1}=s_k+2\hat{k}$, $l_{k+1}=\max\left\{l(t_{k+1}),(l_k+2\hat{k})\right\}$, and $s_{k+1}=s(l_{k+1},t_{k+1})$. Note that by construction, for all $k\in\mathbb{N}$, we have $l_k,s_k\in 2\hat{k}\mathbb{N}$ with $l_{k+1}\geq l_k+1$ and $s_{k+1}>s_k+1$.[17] Furthermore, note that for all $k\in\mathbb{N}$,

$$\mathbb{P}_p\left(\left|\left(C^{\beta}_{B\left([-l_k,l_k]^{d-1}\times[0,t_k]\right)}(D^*)\cap T(l_k,t_k)\right)\right|\geq 2^{d-1}k^*n^*\right)>1-\left(\frac{\epsilon^*}{10}\right)^{2^{d-1}},\text{ and}\tag{A.9a}$$

$$\mathbb{P}_p\left(\left|\left(C^{\beta}_{B\left([-l_k,l_k]^{d-1}\times[0,s_k]\right)}(D^*)\cap T(l_k,s_k)\right)\right|\geq 2^{d-1}k^*n^*\right)\leq 1-\left(\frac{\epsilon^*}{10}\right)^{2^{d-1}}.\tag{A.9b}$$

For all $k\in\mathbb{N}$, define the random variable $N_k:\Omega\longrightarrow\mathbb{N}$ such that for all $\omega\in\Omega$,

$$N_k(\omega)=\left|\left(C^{\beta}_{B\left([-l_k,l_k]^{d-1}\times[0,s_k]\right)}(D^*)\cap\left(T(l_k,s_k)\cup F(l_k,s_k)\right)\right)\right|.$$

Claim A.2. *There exists $k_0\in\mathbb{N}$ such that*

$$\mathbb{P}_p\left(N_{k_0}\geq 2^{d-1}k^*n^*+2(d-1)2^{d-2}n^*\right)>1-\frac{1}{2^{2^{d-1}}}\left(\frac{\epsilon^*}{5}\right)^{d2^{d-1}}.$$

[16]Note that by construction, $s(l(t_1),t_1)>t_1$.
[17]Since $s_{k+1}>t_{k+1}=s_k+2\hat{k}$.

Proof of Claim A.2. For all $k \in \mathbb{N}$, let $\mathcal{G}_k$ be the σ-algebra generated by $\{N_u : 1 \le u \le k\}$. Fix $k \in \mathbb{N}$ and $G \in \mathcal{G}_k$. For a vertex $x \in \big(T(l_k, s_k) \cup F(l_k, s_k)\big)$, define the following event:

$$A_x := \Big\{\omega \in \Omega : \langle x, y\rangle \text{ is closed whenever } y \in \Big(T\big(l_k+1, s_k+1\big) \cup F\big(l_k+1, s_k+1\big)\Big)\Big\}.$$

Clearly, $\mathbb{P}_p(A_x) \ge (1-p)^{2^d}$ for all $x \in \big(T(l_k, s_k) \cup F(l_k, s_k)\big)$. Fix $S \subseteq \big(T(l_k, s_k) \cup F(l_k, s_k)\big)$ such that $|S| < \big[2^{d-1}k^* n^* + 2(d-1)2^{d-2}n^*\big]$. Recall that for all $k \in \mathbb{N}$, we have $l_{k+1} \ge l_k + 1$ and $s_{k+1} > s_k + 1$. It follows that

$$\begin{aligned}
&\mathbb{P}_p\left(N_{k+1} = 0 \;\middle|\; G \text{ and } \left(C^{\beta}_{B\left([-l_k, l_k]^{d-1} \times [0, s_k]\right)}(D^*) \cap \big(T(l_k, s_k) \cup F(l_k, s_k)\big)\right) = S\right) \\
&\ge \mathbb{P}_p\left(\bigcap_{x \in S} A_x\right) \qquad \text{(by the construction of the } \beta\text{-backbend percolation process)} \\
&= \prod_{x \in S} \mathbb{P}_p(A_x) \qquad \text{(since each edge is independent)} \\
&> (1-p)^{2^{2d-1}(k^* n^* + (d-1)n^*)}.
\end{aligned}$$

This, along with Remark A.1, yields

$$\mathbb{P}_p\left(N_{k+1} > 0 \;\middle|\; G \text{ and } N_k < \big[2^{d-1}k^* n^* + 2(d-1)2^{d-2}n^*\big]\right) < 1 - (1-p)^{2^{2d-1}(k^* n^* + (d-1)n^*)}. \quad \text{(A.10)}$$

Since $p < 1$, we have $1 - (1-p)^{2^{2d-1}(k^* n^* + (d-1)n^*)} < 1$. This, together with (A.10) and an argument similar to the one which we use to complete the proof of Claim A.1 from (A.5), yields

$$\mathbb{P}_p\Big(\big|C^{\beta}_{\mathbb{H}}(D^*)\big| = \infty \text{ and } N_k < \big[2^{d-1}k^* n^* + 2(d-1)2^{d-2}n^*\big] \text{ for infinitely many } k \in \mathbb{N}\Big) = 0. \quad \text{(A.11)}$$

The proof of Claim A.2 follows from (A.1) and (A.11) by using an argument similar to the one which we use to obtain (A.8) from Claim A.1. $\square$

By Claim A.2, along with FKG inequality, we have

$$\begin{aligned}
\frac{1}{2^{2^{d-1}}}\left(\frac{\epsilon^*}{5}\right)^{d2^{d-1}} \ge\; &\mathbb{P}_p\left(\left|\left(C^{\beta}_{B\left([-l_{k_0}, l_{k_0}]^{d-1} \times [0, s_{k_0}]\right)}(D^*) \cap T(l_{k_0}, s_{k_0})\right)\right| < 2^{d-1}k^* n^*\right) \\
&\times \mathbb{P}_p\left(\left|\left(C^{\beta}_{B\left([-l_{k_0}, l_{k_0}]^{d-1} \times [0, s_{k_0}]\right)}(D^*) \cap F(l_{k_0}, s_{k_0})\right)\right| < 2(d-1)2^{d-2}n^*\right),
\end{aligned}$$

which, together with (A.9b), yields

$$\left(\frac{\epsilon^*}{5}\right)^{2(d-1)2^{d-2}} > \mathbb{P}_p\left(\left|\left(C^{\beta}_{B\left([-l_{k_0},l_{k_0}]^{d-1}\times[0,s_{k_0}]\right)}(D^*)\cap F(l_{k_0},s_{k_0})\right)\right| < 2(d-1)2^{d-2}n^*\right).$$

By FKG inequality and the construction of the β-backbend percolation process, this implies

$$\mathbb{P}_p\left(\left|\left(C^{\beta}_{B\left([-l_{k_0},l_{k_0}]^{d-1}\times[0,s_{k_0}]\right)}(D^*)\cap F^{v}_{(d-1)^+}(l_{k_0},s_{k_0})\right)\right| < n^*\right) < \frac{\epsilon^*}{5} \text{ for all } v\in\{-1,1\}^{d-2}. \tag{A.12}$$

Furthermore, by the construction of l_{k_0}, t_{k_0}, and s_{k_0}, we have

$$\left(\frac{\epsilon^*}{10}\right)^{2^{d-1}} > \mathbb{P}_p\left(\left|\left(C^{\beta}_{B\left([-l_{k_0},l_{k_0}]^{d-1}\times[0,s_{k_0}-2\hat{k}]\right)}(D^*)\cap T(l_{k_0},s_{k_0}-2\hat{k})\right)\right| < 2^{d-1}k^*n^*\right)$$

By FKG inequality and the construction of the β-backbend percolation process, this implies

$$\mathbb{P}_p\left(\left|\left(C^{\beta}_{B\left([-l_{k_0},l_{k_0}]^{d-1}\times[0,s_{k_0}-2\hat{k}]\right)}(D^*)\cap T^{u}(l_{k_0},s_{k_0}-2\hat{k})\right)\right| < k^*n^*\right) < \frac{\epsilon^*}{10} \text{ for all } u\in\{-1,1\}^{d-1}. \tag{A.13}$$

Claim A.3. *For all $u\in\{-1,1\}^{d-1}$, we have*

$$\mathbb{P}_p\left(\left|\left(C^{\beta}_{B\left([-l_{k_0},l_{k_0}]^{d-1}\times[0,s_{k_0}]\right)}(D^*)\cap T^{u}(l_{k_0},s_{k_0})\right)\right| \geq n^*\right) > 1-\frac{\epsilon^*}{5}.$$

Proof of Claim A.3. Fix $u\in\{-1,1\}^{d-1}$. For a vertex $x\in T^u(l_{k_0},s_{k_0}-2\hat{k})$, define the following event:

$$A_x := \left\{\omega\in\Omega : \begin{array}{l}\text{there is an open path } (x^0=x,\dots,x^{2\hat{k}}) \text{ in } B\left([-l_{k_0},l_{k_0}]^{d-1}\times[0,s_{k_0}]\right) \text{ from } x,\\ \text{where } x^{2i}=x+(0,\dots,0,2i) \text{ for all } i\leq\hat{k}\end{array}\right\}.$$

By construction, $\{A_x\}_{x\in T^u(l_{k_0},s_{k_0}-2\hat{k})}$ are independent events. Furthermore, note that $\mathbb{P}_p(A_x)\geq p^{2\hat{k}}$ for all $x\in T^u(l_{k_0},s_{k_0}-2\hat{k})$. Fix $S\subseteq T^u(l_{k_0},s_{k_0}-2\hat{k})$ such that $|S|\geq k^*n^*$. Make a partition $\{S_1,\dots,S_{n^*}\}$ of S such that $|S_i|\geq k^*$ for all $i=1,\dots,n^*$. For all $i=1,\dots,n^*$, define

$$B_i := \bigcup_{x\in S_i} A_x.$$

Clearly, $\mathbb{P}_p(B_i) \geq 1-(1-p^{2\hat{k}})^{k^*}$ for all $i=1,\ldots,n^*$.[18] Now,

$$\begin{aligned}
&\mathbb{P}_p\left(\underbrace{\left|\left(C^{\beta}_{B\left([-l_{k_0},l_{k_0}]^{d-1}\times[0,s_{k_0}]\right)}(D^*)\cap T^u(l_{k_0},s_{k_0})\right)\right|}_{\geq n^*} \;\middle|\; \underbrace{\left(C^{\beta}_{B\left([-l_{k_0},l_{k_0}]^{d-1}\times[0,s_{k_0}-2\hat{k}]\right)}(D^*)\cap T^u(l_{k_0},s_{k_0}-2\hat{k})\right)}_{=S}\right)\\
&\geq \mathbb{P}_p\left(\bigcap_{i=1}^{n^*} B_i\right) \qquad \text{(by the construction of the } \beta\text{-backbend percolation process)}\\
&\geq \prod_{i=1}^{n^*}\mathbb{P}_p(B_i) \qquad \text{(by FKG inequality)}\\
&\geq \prod_{i=1}^{n^*}\left(1-(1-p^{2\hat{k}})^{k^*}\right)\\
&> 1-\frac{\epsilon^*}{10}. \qquad \text{(by the definition of } k^*\text{)}
\end{aligned}$$

This, together with Remark A.1, implies

$$\begin{aligned}
&\mathbb{P}_p\left(\underbrace{\left|\left(C^{\beta}_{B\left([-l_{k_0},l_{k_0}]^{d-1}\times[0,s_{k_0}]\right)}(D^*)\cap T^u(l_{k_0},s_{k_0})\right)\right|}_{\geq n^*} \;\middle|\; \underbrace{\left|\left(C^{\beta}_{B\left([-l_{k_0},l_{k_0}]^{d-1}\times[0,s_{k_0}-2\hat{k}]\right)}(D^*)\cap T^u(l_{k_0},s_{k_0}-2\hat{k})\right)\right|}_{\geq k^*n^*}\right)\\
&> 1-\frac{\epsilon^*}{10},
\end{aligned}$$

which, along with (A.13), yields

$$\mathbb{P}_p\left(\left|\left(C^{\beta}_{B\left([-l_{k_0},l_{k_0}]^{d-1}\times[0,s_{k_0}]\right)}(D^*)\cap T^u(l_{k_0},s_{k_0})\right)\right|\geq n^*\right) > 1-\frac{\epsilon^*}{5}.$$

This completes the proof of Claim A.3. □

Let $l^*=l_{k_0}$ and $t^*=s_{k_0}$. By construction, we have $l^*,t^*\in 2\hat{k}\mathbb{N}$ with $l^*\geq r^*$.[19] This, along with (A.12) and Claim A.3, completes the proof of Lemma A.1. ■

[18] Note that since $\{A_x\}_{x\in T^u(l_{k_0},s_{k_0}-2\hat{k})}$ are independent events, $\{A^c_x\}_{x\in T^u(l_{k_0},s_{k_0}-2\hat{k})}$ are also independent events.

[19] By construction, $l^*=l_{k_0}\geq l_1\geq r^*$.

A.1.2 Proof of Lemma A.2

Fix an arbitrary $x \in B\big([-l^*, l^*]^{d-2} \times 0 \times 0\big)$. Define the random variable $\tau : \Omega \longrightarrow 2\hat{k}\mathbb{N} \cup \{\infty\}$ such that for all $\omega \in \Omega$,

$$\tau(\omega) := \min \left\{ t \in 2\hat{k}\mathbb{N} : \begin{array}{l} (D^* + z) \subseteq C^{\beta}_{B\left([-2l^*, 2l^*]^{d-2} \times [-l^*, l^* + 2r^*] \times [0,t]\right)}(D^* + x) \\ \text{for some } z \in B\big([-l^*, l^*]^{d-2} \times (l^* + r^*) \times t\big) \end{array} \right\},$$

where $\min \emptyset = \infty$.

Claim A.4. $\mathbb{P}_p\Big(\tau \in [0, s^*]\Big) > \Big(1 - \frac{\epsilon^*}{5}\Big)^2$.

Proof of Claim A.4. Choose $v \in \{-1, 1\}^{d-2}$ such that $x_i v_i \leq 0$ for all $i = 1, \ldots, d-2$.[20] Define the following events:

$$A_1 := \left\{ \omega \in \Omega : \begin{array}{l} \exists\, S_1 \subseteq \Big(C^{\beta}_{\left(B\left([-l^*, l^*]^{d-1} \times [0,t^*]\right) + x\right)}(D^* + x) \cap \big(F^v_{(d-1)^+}(l^*, t^*) + x\big)\Big) \text{ with } |S_1| \geq m^* \\ \text{such that for all distinct } y_1, y_2 \in S_1, L^{\infty}(y_1, y_2) \geq 4r^* + 4\hat{k} + 1 \end{array} \right\},$$

$$A_2 := \left\{ \omega \in \Omega : \begin{array}{l} \exists\, y \in \Big(C^{\beta}_{\left(B\left([-l^*, l^*]^{d-1} \times [0,t^*]\right) + x\right)}(D^* + x) \cap \big(F^v_{(d-1)^+}(l^*, t^*) + x\big)\Big) \text{ such that} \\ \Big(B\big([-r^*, r^*]^{d-2} \times [0, 2r^*] \times [2r^*, 2r^* + 2\hat{k}]\big) + y\Big) \subseteq C^{0}_{\left(B\left([-r^*, r^*]^{d-2} \times [0,2r^*] \times [0, 2r^* + 2\hat{k}]\right) + y\right)}(\{y\}) \end{array} \right\}.$$

By the construction of v, $\big(F^v_{(d-1)^+}(l^*, t^*) + x\big) \subseteq B\big([-l^*, l^*]^{d-2} \times l^* \times [0, t^*]\big)$. Since $2r^* \geq \max\{\beta_0, \ldots, \beta_{\hat{k}-1}\}$, $s^* = t^* + 2r^* + 2\hat{k}$, and $\big(F^v_{(d-1)^+}(l^*, t^*) + x\big) \subseteq B\big([-l^*, l^*]^{d-2} \times l^* \times [0, t^*]\big)$, it follows from the construction of A_2 that $A_2 \subseteq \big\{\omega \in \Omega : \tau(\omega) \in [0, s^*]\big\}$. So, we have

$$\begin{aligned}
\mathbb{P}_p\Big(\tau \in [0, s^*]\Big) &\geq \mathbb{P}_p\Big(A_2\Big) \\
&\geq \mathbb{P}_p\Big(A_2 \mid A_1\Big) \times \mathbb{P}_p\Big(A_1\Big) \\
&\geq \mathbb{P}_p\Big(A_2 \mid A_1\Big) \\
&\quad \times \mathbb{P}_p\bigg(A_1 \;\bigg|\; \Big|\Big(C^{\beta}_{\left(B\left([-l^*, l^*]^{d-1} \times [0,t^*]\right) + x\right)}(D^* + x) \cap \big(F^v_{(d-1)^+}(l^*, t^*) + x\big)\Big)\Big| \geq n^*\bigg) \\
&\quad \times \mathbb{P}_p\bigg(\Big|\Big(C^{\beta}_{\left(B\left([-l^*, l^*]^{d-1} \times [0,t^*]\right) + x\right)}(D^* + x) \cap \big(F^v_{(d-1)^+}(l^*, t^*) + x\big)\Big)\Big| \geq n^*\bigg).
\end{aligned} \tag{A.14}$$

Define the mapping $S_1 : A_1 \longrightarrow \mathcal{P}(\mathbb{H})$ such that for all $\omega \in A_1$,

[20] Note that there may be more than one such v.

(i) $S_1(\omega) \subseteq \Big(C^{\beta}_{\left(B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)+x\right)}(D^*+x) \cap \big(F^v_{(d-1)^+}(l^*,t^*)+x\big)\Big)$, and

(ii) $|S_1(\omega)| \geq m^*$, where $L^\infty(y_1,y_2) \geq 4r^* + 4\hat{k} + 1$ for all distinct $y_1, y_2 \in S_1(\omega)$.

By the definition of m^*, A_1, S_1 and A_2, along with Remark A.1 and the construction of the β-backbend percolation process, we have

$$\begin{aligned}
&\mathbb{P}_p\Big(A_2 \mid A_1\Big) \\
&\geq \mathbb{P}_p\left(\begin{matrix}\exists\, y \in S_1 \text{ such that} \\ \Big(B\big([-r^*,r^*]^{d-2}\times[0,2r^*]\times[2r^*,2r^*+2\hat{k}]\big)+y\Big) \subseteq C^0_{\left(B\left([-r^*,r^*]^{d-2}\times[0,2r^*]\times[0,2r^*+2\hat{k}]\right)+y\right)}(\{y\})\end{matrix} \;\middle|\; A_1\right) \\
&> 1-\frac{\epsilon^*}{5}.
\end{aligned} \tag{A.15}$$

By the definition of A_1 and n^*,

$$\mathbb{P}_p\left(A_1 \;\middle|\; \Big|\Big(C^{\beta}_{\left(B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)+x\right)}(D^*+x) \cap \big(F^v_{(d-1)^+}(l^*,t^*)+x\big)\Big)\Big| \geq n^*\right) = 1, \tag{A.16}$$

and by Lemma A.1 and the construction of the β-backbend percolation process,

$$\mathbb{P}_p\left(\Big|\Big(C^{\beta}_{\left(B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)+x\right)}(D^*+x) \cap \big(F^v_{(d-1)^+}(l^*,t^*)+x\big)\Big)\Big| \geq n^*\right) > 1-\frac{\epsilon^*}{5}.$$

This, together with (A.14), (A.15), and (A.16), completes the proof of Claim A.4. □

Define the mapping $\zeta : \Omega \longrightarrow B\big([-l^*,l^*]^{d-2}\times(l^*+r^*)\times 2\hat{k}\mathbb{N}\big) \cup \{\infty\}$ such that for all $\omega \in \Omega$,

$$\zeta(\omega) := \begin{cases} z & \text{if } \tau(\omega) < \infty \text{ and} \\ & (D^*+z) \subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-l^*,l^*+2r^*]\times[0,z_d]\right)}(D^*+x) \text{ with } z_d = \tau(\omega); \\ \infty & \text{if } \tau(\omega) = \infty \end{cases}$$

Claim A.5.

$$\mathbb{P}_p\left(\begin{matrix}(D^*+z) \subseteq C^{\beta}_{\left(B\left([-l^*,l^*]^{d-1}\times[0,s^*]\right)+\zeta\right)}(D^*+\zeta) \\ \textit{for some } z \in B\big([-l^*,l^*]^{d-2}\times[l^*,2l^*]\times(\tau+s^*)\big)\end{matrix} \;\middle|\; \tau \in [0,s^*]\right) > \left(1-\frac{\epsilon^*}{5}\right)^2.$$

Proof of Claim A.5. Define the mapping $u : \Omega \longrightarrow \{-1,1\}^{d-1} \cup \{\infty\}$ as follows.

(i) For all $\omega \in \Omega$ with $\zeta(\omega) \in \mathbb{H}$, $u_{d-1}(\omega) = 1$ and $\zeta_i(\omega)u_i(\omega) \leq 0$ for all $i = 1, \dots, d-2$.

(ii) For all $\omega \in \Omega$ with $\zeta(\omega) = \infty$, $u(\omega) = \infty$.

Since β is $\hat{k}$-cyclic, it follows from the definition of τ and ζ, the construction of the β-backbend percolation process, Lemma A.1, and Remark A.1 that

$$\mathbb{P}_p\left(\left|\left(C^{\beta}_{\left(B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)+\zeta\right)}(D^*+\zeta)\cap\left(T^u(l^*,t^*)+\zeta\right)\right)\right| \geq n^* \;\middle|\; \tau \in [0,s^*]\right) > 1 - \frac{\epsilon^*}{5}. \tag{A.17}$$

Define the following event:

$$B_1 := \left\{\omega \in \Omega : \begin{array}{l} \exists\, S_2 \subseteq \left(C^{\beta}_{\left(B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)+\zeta\right)}(D^*+\zeta)\cap\left(T^u(l^*,t^*)+\zeta\right)\right) \text{ with } |S_2| \geq m^* \\ \text{such that for all distinct } y_1, y_2 \in S_2, L^{\infty}(y_1,y_2) \geq 4r^* + 4\hat{k} + 1 \end{array}\right\}.$$

Now, we have

$$\begin{aligned} &\mathbb{P}_p\left(B_1 \;\middle|\; \tau \in [0,s^*]\right) \\ \geq\, &\mathbb{P}_p\left(B_1 \;\middle|\; \tau \in [0,s^*] \text{ and } \left|\left(C^{\beta}_{\left(B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)+\zeta\right)}(D^*+\zeta)\cap\left(T^u(l^*,t^*)+\zeta\right)\right)\right| \geq n^*\right) \\ &\times \mathbb{P}_p\left(\left|\left(C^{\beta}_{\left(B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)+\zeta\right)}(D^*+\zeta)\cap\left(T^u(l^*,t^*)+\zeta\right)\right)\right| \geq n^* \;\middle|\; \tau \in [0,s^*]\right). \end{aligned} \tag{A.18}$$

By the definition of B_1 and n^*, together with Remark A.1, we have

$$\mathbb{P}_p\left(B_1 \;\middle|\; \tau \in [0,s^*] \text{ and } \left|\left(C^{\beta}_{\left(B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)+\zeta\right)}(D^*+\zeta)\cap\left(T^u(l^*,t^*)+\zeta\right)\right)\right| \geq n^*\right) = 1.$$

This, together with (A.17) and (A.18), implies

$$\mathbb{P}_p\left(B_1 \;\middle|\; \tau \in [0,s^*]\right) > 1 - \frac{\epsilon^*}{5}. \tag{A.19}$$

Define the mapping $S_2 : B_1 \longrightarrow \mathcal{P}(\mathbb{H})$ such that for all $\omega \in B_1$,

(i) $S_2(\omega) \subseteq \left(C^{\beta}_{\left(B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)+\zeta\right)}(D^*+\zeta)\cap\left(T^u(l^*,t^*)+\zeta\right)\right)$, and

(ii) $|S_2(\omega)| \geq m^*$, where $L^{\infty}(y_1,y_2) \geq 4r^* + 4\hat{k} + 1$ for all distinct $y_1, y_2 \in S_2(\omega)$.

By the definition of τ, ζ, u, and B_1, we have $\left(T^u(l^*,t^*)+\zeta\right) \subseteq B\left([-l^*,l^*]^{d-2}\times[l^*+r^*, 2l^*+r^*]\times(\tau + t^*)\right)$ for all $\omega \in B_1$. Define the mappings $\hat{S}_2, \bar{S}_2 : B_1 \longrightarrow \mathcal{P}(\mathbb{H})$ such that for all $\omega \in B_1$, $\hat{S}_2(\omega) := \{y \in$

$S_2(\omega) : y_{d-1} > 2l^*\}$, and $\bar{S}_2(\omega) := \{y \in S_2(\omega) : y_{d-1} \leq 2l^*\}$. Define the following sub-events of B_1:

$$\hat{B}_1 := \left\{ \omega \in B_1 : \begin{array}{l} \exists\, y \in \hat{S}_2(\omega) \text{ such that} \\ \left(D^* + y + (0, \ldots, 0, -r^*, 2r^* + 2\hat{k})\right) \subseteq C^0_{\left(B\left([-r^*,r^*]^{d-2} \times [-2r^*,0] \times [0,2r^*+2\hat{k}]\right)+y\right)}(\{y\}) \end{array} \right\}, \text{and}$$

$$\bar{B}_1 := \left\{ \omega \in B_1 : \begin{array}{l} \exists\, y \in \bar{S}_2(\omega) \text{ such that} \\ \left(D^* + y + (0, \ldots, 0, 2r^* + 2\hat{k})\right) \subseteq C^0_{\left(B\left([-r^*,r^*]^{d-1} \times [0,2r^*+2\hat{k}]\right)+y\right)}(\{y\}) \end{array} \right\}.$$

By the definition of m^*, B_1, $\hat{B}_1$, and $\bar{B}_1$, along with Remark A.1 and the construction of the β-backbend percolation process, we have

$$\mathbb{P}_p\left(\hat{B}_1 \cup \bar{B}_1 \;\middle|\; B_1 \text{ and } \tau \in [0, s^*]\right) > 1 - \frac{\epsilon^*}{5}. \tag{A.20}$$

Since $l^* \geq r^*$ (see Lemma A.1 for details) and $s^* = t^* + 2r^* + 2\hat{k}$, we have

$$\begin{aligned}
&\mathbb{P}_p\left(\begin{array}{l} (D^* + z) \subseteq C^\beta_{\left(B\left([-l^*,l^*]^{d-1} \times [0,s^*]\right)+\zeta\right)}(D^* + \zeta) \\ \text{for some } z \in B\left([-l^*, l^*]^{d-2} \times [l^*, 2l^*] \times (\tau + s^*)\right) \end{array} \;\middle|\; \tau \in [0, s^*] \right) \\
\geq\ &\mathbb{P}_p\left(\begin{array}{l} (D^* + z) \subseteq C^\beta_{\left(B\left([-l^*,l^*]^{d-1} \times [0,s^*]\right)+\zeta\right)}(D^* + \zeta) \\ \text{for some } z \in B\left([-l^*, l^*]^{d-2} \times [l^*, 2l^*] \times (\tau + s^*)\right) \end{array} \;\middle|\; B_1 \text{ and } \tau \in [0, s^*] \right) \times \mathbb{P}_p\left(B_1 \;\middle|\; \tau \in [0, s^*]\right) \\
\geq\ &\mathbb{P}_p\left(\hat{B}_1 \cup \bar{B}_1 \;\middle|\; B_1 \text{ and } \tau \in [0, s^*]\right) \times \mathbb{P}_p\left(B_1 \;\middle|\; \tau \in [0, s^*]\right).
\end{aligned}$$

This, together with (A.19) and (A.20), completes the proof of Claim A.5. $\square$

By A_x, we denote the event $\left\{\omega \in \Omega : (D^* + z) \subseteq C^\beta_{B\left([-2l^*,2l^*]^{d-2} \times [-l^*,3l^*] \times [0,z_d]\right)}(D^* + x) \text{ for some } z \in B\left([-l^*, l^*]^{d-2} \times [l^*, 2l^*] \times [s^*, 2s^*]\right) \text{ with } z_d \in 2\hat{k}\mathbb{N}\right\}$. Since $l^* \geq r^*$, by the definition of τ, ζ, and A_x, the construction of the lattice structure, and the construction of the β-backbend percolation process, we have

$$
\begin{aligned}
&\mathbb{P}_p(A_x) \\
\geq\ &\mathbb{P}_p\Big(A_x \ \Big|\ \tau \in [0,s^*]\Big) \times \mathbb{P}_p\Big(\tau \in [0,s^*]\Big) \\
\geq\ &\mathbb{P}_p\left(\begin{array}{l} (D^*+z) \subseteq C^{\beta}_{\left(B\left([-l^*,l^*]^{d-1}\times[0,s^*]\right)+\zeta\right)}(D^*+\zeta) \\ \text{for some } z \in B\big([-l^*,l^*]^{d-2}\times[l^*,2l^*]\times(\tau+s^*)\big) \end{array}\ \middle|\ \tau\in[0,s^*]\right) \times \mathbb{P}_p\Big(\tau \in [0,s^*]\Big) \qquad \text{(A.21)} \\
>\ &\left(1-\frac{\epsilon^*}{5}\right)^4 \qquad \text{(by Claim A.4 and Claim A.5)} \\
>\ &1-\epsilon^*.
\end{aligned}
$$

Note that the event A_x depends only on the edges in the set $\Big(B\big([-2l^*,2l^*]^{d-2}\times[-l^*,3l^*]\times[0,2s^*]\big)\Big)^2$. Since this set is finite, it follows that $\mathbb{P}_p(A_x)$ is a continuous function of p. Moreover, by construction, $\mathbb{P}_p(A_x)$ is an increasing function of p. Since $\mathbb{P}_p(A_x)$ is an increasing continuous function of p, (A.21) implies there exists $\delta_x > 0$ with $\delta_x < p$ such that $\mathbb{P}_{p-\delta_x}(A_x) > 1-\epsilon^*$.

Let

$$\delta := \min\ \Big\{\delta_x : x \in B\big([-l^*,l^*]^{d-2}\times 0\times 0\big)\Big\}.$$

Since $B\big([-l^*,l^*]^{d-2}\times 0\times 0\big)$ is a finite set of vertices, by the construction of δ, we have $0 < \delta < p$. Furthermore, for all $x \in B\big([-l^*,l^*]^{d-2}\times 0\times 0\big)$, since $\mathbb{P}_p(A_x)$ is an increasing continuous function of p and $\mathbb{P}_{p-\delta_x}(A_x) > 1-\epsilon^*$, it follows from the construction of δ that $\mathbb{P}_{p-\delta}(A_x) > 1-\epsilon^*$ for all $x \in B\big([-l^*,l^*]^{d-2}\times 0\times 0\big)$. This completes the proof of Lemma A.2. ∎

A.1.3 Proof of Lemma A.3

Since β is $\hat{k}$-cyclic, Lemma A.2, together with the construction of the β-backbend percolation process, implies that for all $y \in B\big([-l^*,l^*]^{d-2}\times\mathbb{Z}\times\mathbb{Z}_+\big)$ with $y_d \in 2\hat{k}\mathbb{Z}_+$,

$$
\mathbb{P}_{p-\delta^*}\left(\begin{array}{l} (D^*+z) \subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-l^*+y_{d-1},3l^*+y_{d-1}]\times[y_d,z_d]\right)}(D^*+y) \\ \text{for some } z \in B\big([-l^*,l^*]^{d-2}\times[l^*+y_{d-1},2l^*+y_{d-1}]\times[s^*+y_d,2s^*+y_d]\big) \\ \text{with } z_d \in 2\hat{k}\mathbb{N} \end{array}\right) > 1-\epsilon^*, \qquad \text{(A.22a)}
$$

$$\mathbb{P}_{p-\delta^*}\left(\begin{array}{l}(D^*+z)\subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-3l^*+y_{d-1},l^*+y_{d-1}]\times[y_d,z_d]\right)}(D^*+y)\\ \text{for some } z\in B\left([-l^*,l^*]^{d-2}\times[-2l^*+y_{d-1},-l^*+y_{d-1}]\times[s^*+y_d,2s^*+y_d]\right)\\ \text{with } z_d\in 2\hat{k}\mathbb{N}\end{array}\right)>1-\epsilon^*. \tag{A.22b}$$

Fix an arbitrary $x\in B\left([-l^*,l^*]^{d-2}\times[-2l^*,2l^*]\times[0,2s^*]\right)$ with $x_d\in 2\hat{k}\mathbb{Z}_+$. By A_x, we denote the event $\Big\{\omega\in\Omega : (D^*+z)\subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-3l^*,4l^*]\times[x_d,z_d]\right)}(D^*+x)$ for some $z\in B\big([-l^*,l^*]^{d-2}\times[-l^*,3l^*]\times[2s^*,4s^*]\big)$ with $z_d\in 2\hat{k}\mathbb{N}\Big\}$. To prove Lemma A.3, we distinguish the following five cases.

Case 1: Suppose $x\in B\left([-l^*,l^*]^{d-2}\times[-2l^*,l^*]\times[s^*,2s^*]\right)$.
It follows from (A.22a) that $\mathbb{P}_{p-\delta^*}(A_x)>1-\epsilon^*$.

Case 2: Suppose $x\in B\left([-l^*,l^*]^{d-2}\times[l^*,2l^*]\times[s^*,2s^*]\right)$.
It follows from (A.22b) that $\mathbb{P}_{p-\delta^*}(A_x)>1-\epsilon^*$.

Case 3: Suppose $x\in B\left([-l^*,l^*]^{d-2}\times[-2l^*,0]\times[0,s^*]\right)$.
It follows from (A.22a) that

$$\mathbb{P}_{p-\delta^*}\left(\begin{array}{l}(D^*+z)\subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-3l^*,3l^*]\times[x_d,z_d]\right)}(D^*+x)\\ \text{for some } z\in B\left([-l^*,l^*]^{d-2}\times[-l^*,2l^*]\times[s^*,3s^*]\right)\text{ with } z_d\in 2\hat{k}\mathbb{N}\end{array}\right)>1-\epsilon^*. \tag{A.23}$$

Define the following events:

$$A_1:=\left\{\omega\in\Omega:\begin{array}{l}(D^*+z)\subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-3l^*,3l^*]\times[x_d,z_d]\right)}(D^*+x)\\ \text{for some } z\in B\left([-l^*,l^*]^{d-2}\times[-l^*,2l^*]\times[2s^*,3s^*]\right)\text{ with } z_d\in 2\hat{k}\mathbb{N}\end{array}\right\},$$

$$A_2:=\left\{\omega\in\Omega:\begin{array}{l}(D^*+z)\subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-3l^*,3l^*]\times[x_d,z_d]\right)}(D^*+x)\\ \text{for some } z\in B\left([-l^*,l^*]^{d-2}\times[-l^*,l^*]\times[s^*,2s^*]\right)\text{ with } z_d\in 2\hat{k}\mathbb{N}\end{array}\right\}\setminus A_1,\text{ and}$$

$$A_3:=\left\{\omega\in\Omega:\begin{array}{l}(D^*+z)\subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-3l^*,3l^*]\times[x_d,z_d]\right)}(D^*+x)\\ \text{for some } z\in B\left([-l^*,l^*]^{d-2}\times[l^*,2l^*]\times[s^*,2s^*]\right)\text{ with } z_d\in 2\hat{k}\mathbb{N}\end{array}\right\}\setminus(A_1\cup A_2).$$

Clearly A_1, A_2, and A_3 are disjoint. By the construction of A_1, A_2, and A_3, (A.23) implies $\mathbb{P}_{p-\delta^*}\Big(A_1\cup A_2\cup A_3\Big)>1-\epsilon^*$. Since A_1, A_2, and A_3 are disjoint, $\mathbb{P}_{p-\delta^*}\Big(A_1\cup A_2\cup A_3\Big)>1-\epsilon^*$ implies

$$\mathbb{P}_{p-\delta^*}(A_1)+\mathbb{P}_{p-\delta^*}(A_2)+\mathbb{P}_{p-\delta^*}(A_3)>1-\epsilon^*. \tag{A.24}$$

Define the mappings $z' : A_2 \longrightarrow B\big([-l^*,l^*]^{d-2} \times [-l^*,l^*] \times [s^*,2s^*]\big)$ and $z'' : A_3 \longrightarrow B\big([-l^*,l^*]^{d-2} \times [l^*,2l^*] \times [s^*,2s^*]\big)$ as follows.

(i) For all $\omega \in A_2$, $\big(D^* + z'(\omega)\big) \subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-3l^*,3l^*]\times[x_d,z'_d(\omega)]\right)}(D^*+x)$ with $z'_d(\omega) \in 2\hat{k}\mathbb{N}$.

(ii) For all $\omega \in A_3$, $\big(D^* + z''(\omega)\big) \subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-3l^*,3l^*]\times[x_d,z''_d(\omega)]\right)}(D^*+x)$ with $z''_d(\omega) \in 2\hat{k}\mathbb{N}$.

By the definition of z' and z'', along with Remark A.1, (A.22a), and (A.22b), we have

$$\mathbb{P}_{p-\delta^*}\left(\begin{array}{l}(D^*+z) \subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-3l^*,4l^*]\times[z'_d,z_d]\right)}(D^*+z') \\ \text{for some } z \in B\big([-l^*,l^*]^{d-2}\times[-l^*,3l^*]\times[2s^*,4s^*]\big) \text{ with } z_d \in 2\hat{k}\mathbb{N}\end{array} \;\middle|\; A_2\right) > 1-\epsilon^*, \text{ and}$$
$$\mathbb{P}_{p-\delta^*}\left(\begin{array}{l}(D^*+z) \subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-3l^*,4l^*]\times[z''_d,z_d]\right)}(D^*+z'') \\ \text{for some } z \in B\big([-l^*,l^*]^{d-2}\times[-l^*,3l^*]\times[2s^*,4s^*]\big) \text{ with } z_d \in 2\hat{k}\mathbb{N}\end{array} \;\middle|\; A_3\right) > 1-\epsilon^*. \tag{A.25}$$

So, we have

$$\begin{aligned}
&\mathbb{P}_{p-\delta^*}(A_x) \\
\geq\ &\mathbb{P}_{p-\delta^*}\Big(A_x \cap \big(A_1 \cup A_2 \cup A_3\big)\Big) \\
=\ &\mathbb{P}_{p-\delta^*}(A_x \cap A_1) + \mathbb{P}_{p-\delta^*}(A_x \cap A_2) + \mathbb{P}_{p-\delta^*}(A_x \cap A_3) \qquad \text{(since } A_1, A_2, \text{ and } A_3 \text{ are disjoint)} \\
=\ &\mathbb{P}_{p-\delta^*}(A_1) + \Big(\mathbb{P}_{p-\delta^*}(A_x \mid A_2) \times \mathbb{P}_{p-\delta^*}(A_2)\Big) + \Big(\mathbb{P}_{p-\delta^*}(A_x \mid A_3) \times \mathbb{P}_{p-\delta^*}(A_3)\Big) \qquad \text{(since } A_1 \subseteq A_x) \\
\geq\ &\mathbb{P}_{p-\delta^*}(A_1) + \\
&\left[\mathbb{P}_{p-\delta^*}\left(\begin{array}{l}(D^*+z) \subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-3l^*,4l^*]\times[z'_d,z_d]\right)}(D^*+z') \\ \text{for some } z \in B\big([-l^*,l^*]^{d-2}\times[-l^*,3l^*]\times[2s^*,4s^*]\big) \text{ with } z_d \in 2\hat{k}\mathbb{N}\end{array} \;\middle|\; A_2\right) \times \mathbb{P}_{p-\delta^*}(A_2)\right] + \\
&\left[\mathbb{P}_{p-\delta^*}\left(\begin{array}{l}(D^*+z) \subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-3l^*,4l^*]\times[z''_d,z_d]\right)}(D^*+z'') \\ \text{for some } z \in B\big([-l^*,l^*]^{d-2}\times[-l^*,3l^*]\times[2s^*,4s^*]\big) \text{ with } z_d \in 2\hat{k}\mathbb{N}\end{array} \;\middle|\; A_3\right) \times \mathbb{P}_{p-\delta^*}(A_3)\right] \\
>\ &\mathbb{P}_{p-\delta^*}(A_1) + \Big((1-\epsilon^*) \times \mathbb{P}_{p-\delta^*}(A_2)\Big) + \Big((1-\epsilon^*) \times \mathbb{P}_{p-\delta^*}(A_3)\Big) \qquad \text{(by (A.25))} \\
>\ &(1-\epsilon^*) \times \Big(\mathbb{P}_{p-\delta^*}(A_1) + \mathbb{P}_{p-\delta^*}(A_2) + \mathbb{P}_{p-\delta^*}(A_3)\Big) \\
>\ &(1-\epsilon^*)^2. \qquad \text{(by (A.24))}
\end{aligned}$$

Case 4: Suppose $x \in B\big([-l^*,l^*]^{d-2} \times [0,l^*] \times [0,s^*]\big)$.

It follows from (A.22a) that

$$\mathbb{P}_{p-\delta^*}\left(\begin{array}{l}(D^*+z)\subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-l^*,4l^*]\times[x_d,z_d]\right)}(D^*+x)\\ \text{for some } z\in B\left([-l^*,l^*]^{d-2}\times[l^*,3l^*]\times[s^*,3s^*]\right) \text{ with } z_d\in 2\hat{k}\mathbb{N}\end{array}\right)>1-\epsilon^*. \tag{A.26}$$

Define the following events:

$$A_1:=\left\{\omega\in\Omega:\begin{array}{l}(D^*+z)\subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-l^*,4l^*]\times[x_d,z_d]\right)}(D^*+x)\\ \text{for some } z\in B\left([-l^*,l^*]^{d-2}\times[l^*,3l^*]\times[2s^*,3s^*]\right) \text{ with } z_d\in 2\hat{k}\mathbb{N}\end{array}\right\}, \text{ and}$$

$$A_2:=\left\{\omega\in\Omega:\begin{array}{l}(D^*+z)\subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-l^*,4l^*]\times[x_d,z_d]\right)}(D^*+x)\\ \text{for some } z\in B\left([-l^*,l^*]^{d-2}\times[l^*,3l^*]\times[s^*,2s^*]\right) \text{ with } z_d\in 2\hat{k}\mathbb{N}\end{array}\right\}\setminus A_1.$$

Clearly A_1 and A_2 are disjoint. By the construction of A_1 and A_2, (A.26) implies $\mathbb{P}_{p-\delta^*}\left(A_1\cup A_2\right)>1-\epsilon^*$. Since A_1 and A_2 are disjoint, $\mathbb{P}_{p-\delta^*}\left(A_1\cup A_2\right)>1-\epsilon^*$ implies

$$\mathbb{P}_{p-\delta^*}(A_1)+\mathbb{P}_{p-\delta^*}(A_2)>1-\epsilon^*. \tag{A.27}$$

Define the mapping $z':A_2\longrightarrow B\left([-l^*,l^*]^{d-2}\times[l^*,3l^*]\times[s^*,2s^*]\right)$ such that for all $\omega\in A_2$, we have $\left(D^*+z'(\omega)\right)\subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-l^*,4l^*]\times[x_d,z'_d(\omega)]\right)}(D^*+x)$ with $z'_d(\omega)\in 2\hat{k}\mathbb{N}$. By the definition of z', together with Remark A.1 and (A.22b), we have

$$\mathbb{P}_{p-\delta^*}\left(\begin{array}{l}(D^*+z)\subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-3l^*,4l^*]\times[z'_d,z_d]\right)}(D^*+z')\\ \text{for some } z\in B\left([-l^*,l^*]^{d-2}\times[-l^*,3l^*]\times[2s^*,4s^*]\right) \text{ with } z_d\in 2\hat{k}\mathbb{N}\end{array}\;\middle|\; A_2\right)>1-\epsilon^*. \tag{A.28}$$

Since $A_1\subseteq A_x$, and A_1 and A_2 are disjoint, (A.27) and (A.28), along with an argument similar to the one which we use to complete the proof for Case 3 of this lemma from (A.25), together imply $\mathbb{P}_{p-\delta^*}\left(A_x\right)>\left(1-\epsilon^*\right)^2$.

Case 5: Suppose $x\in B\left([-l^*,l^*]^{d-2}\times[l^*,2l^*]\times[0,s^*]\right)$.

It follows from (A.22b) that

$$\mathbb{P}_{p-\delta^*}\left(\begin{array}{l}(D^*+z)\subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-2l^*,3l^*]\times[x_d,z_d]\right)}(D^*+x)\\ \text{for some } z\in B\left([-l^*,l^*]^{d-2}\times[-l^*,l^*]\times[s^*,3s^*]\right) \text{ with } z_d\in 2\hat{k}\mathbb{N}\end{array}\right)>1-\epsilon^*. \tag{A.29}$$

Define the following events:

$$A_1 := \left\{ \omega \in \Omega : \begin{array}{l} (D^* + z) \subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-2l^*,3l^*]\times[x_d,z_d]\right)}(D^* + x) \\ \text{for some } z \in B\big([-l^*,l^*]^{d-2} \times [-l^*,l^*] \times [2s^*,3s^*]\big) \text{ with } z_d \in 2\hat{k}\mathbb{N} \end{array} \right\}, \text{ and}$$

$$A_2 := \left\{ \omega \in \Omega : \begin{array}{l} (D^* + z) \subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-2l^*,3l^*]\times[x_d,z_d]\right)}(D^* + x) \\ \text{for some } z \in B\big([-l^*,l^*]^{d-2} \times [-l^*,l^*] \times [s^*,2s^*]\big) \text{ with } z_d \in 2\hat{k}\mathbb{N} \end{array} \right\} \setminus A_1.$$

Clearly A_1 and A_2 are disjoint. By the construction of A_1 and A_2, (A.29) implies $\mathbb{P}_{p-\delta^*}\big(A_1 \cup A_2\big) > 1 - \epsilon^*$. Since A_1 and A_2 are disjoint, $\mathbb{P}_{p-\delta^*}\big(A_1 \cup A_2\big) > 1 - \epsilon^*$ implies

$$\mathbb{P}_{p-\delta^*}(A_1) + \mathbb{P}_{p-\delta^*}(A_2) > 1 - \epsilon^*. \tag{A.30}$$

Define the mapping $z' : A_2 \longrightarrow B\big([-l^*,l^*]^{d-2} \times [-l^*,l^*] \times [s^*,2s^*]\big)$ such that for all $\omega \in A_2$, we have $\big(D^* + z'(\omega)\big) \subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-2l^*,3l^*]\times[x_d,z'_d(\omega)]\right)}(D^* + x)$ with $z'_d(\omega) \in 2\hat{k}\mathbb{N}$. By the definition of z', together with Remark A.1 and (A.22a), we have

$$\mathbb{P}_{p-\delta^*}\left(\begin{array}{l} (D^* + z) \subseteq C^{\beta}_{B\left([-2l^*,2l^*]^{d-2}\times[-3l^*,4l^*]\times[z'_d,z_d]\right)}(D^* + z') \\ \text{for some } z \in B\big([-l^*,l^*]^{d-2} \times [-l^*,3l^*] \times [2s^*,4s^*]\big) \text{ with } z_d \in 2\hat{k}\mathbb{N} \end{array} \;\middle|\; A_2 \right) > 1 - \epsilon^*. \tag{A.31}$$

Since $A_1 \subseteq A_x$, and A_1 and A_2 are disjoint, (A.30) and (A.31), along with an argument similar to the one which we use to complete the proof for Case 3 of this lemma from (A.25), together imply $\mathbb{P}_{p-\delta^*}\big(A_x\big) > \big(1 - \epsilon^*\big)^2$.

This completes the proof of Lemma A.3. ■

A.1.4 Proof of Lemma A.4

Fix $i \in \mathbb{N}$ and $x \in B\big([-l^*,l^*]^{d-2} \times [-2l^*,2l^*] \times [0,2s^*]\big)$ with $x_d \in 2\hat{k}\mathbb{Z}_+$. For all $z \in B_i^+$, define

$$\mathcal{S}^+(z) := \left(\bigcup_{j=0}^{i-1} B\big([-2l^*,2l^*]^{d-2} \times [jl^* - 3l^*, jl^* + 4l^*] \times [2js^*, 2js^* + 4s^*]\big) \right) \cap \mathcal{R}^+(z).$$

Since β is $\hat{k}$-cyclic, using the result in Lemma A.3 repeatedly, it follows that

$$\mathbb{P}_{p-\delta^*}\Big((D^* + z) \subseteq C^{\beta}_{\mathcal{S}^+(z)}(D^* + x) \text{ for some } z \in B_i^+ \text{ with } z_d \in 2\hat{k}\mathbb{N} \Big) > (1 - \epsilon^*)^{2i}. \tag{A.32}$$

By the definition of $\mathcal{S}^+(z)$, we have $\mathcal{S}^+(z) \subseteq \mathcal{R}^+(z)$ for all $z \in B_i^+$. This, together with (A.32), implies $\mathbb{P}_{p-\delta^*}\big(G_i^+(x)\big) > (1-\epsilon^*)^{2i}$ which completes the proof of Lemma A.4. ■

A.1.5 Proof of Lemma A.6

Using a similar argument as for Claim A.2 in Lemma A.1, it follows from (A.4) that there exists $l^* \in 2\hat{k}\mathbb{N}$ with $l^* \geq r^*$ such that

$$\mathbb{P}_p\left(\left|\left(C^\beta_{B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)}(D^*) \cap F(l^*,t^*)\right)\right| \geq 2(d-1)2^{d-2}n^*\right) > 1 - \left(\frac{\epsilon^*}{2}\right)^{(d-1)2^{d-1}},$$

which, in particular, means

$$\mathbb{P}_p\left(\left|\left(C^\beta_{B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)}(D^*) \cap F(l^*,t^*)\right)\right| < 2(d-1)2^{d-2}n^*\right) < \left(\frac{\epsilon^*}{2}\right)^{2(d-1)2^{d-2}}.$$

By FKG inequality and the construction of the β-backbend percolation process, this implies that

$$\mathbb{P}_p\left(\left|\left(C^\beta_{B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)}(D^*) \cap F^v_{(d-1)^+}(l^*,t^*)\right)\right| < n^*\right) < \frac{\epsilon^*}{2} \text{ for all } v \in \{-1,1\}^{d-2},$$

which completes the proof of Lemma A.6. ■

A.1.6 Proof of Lemma A.7

Fix an arbitrary $x \in B\big([-l^*,l^*]^{d-2}\times 0\times 0\big)$. By A_x, we denote the event $\Big\{\omega \in \Omega : (D^*+z) \subseteq C^\beta_{B\left([-2l^*,2l^*]^{d-2}\times[-l^*,3l^*]\times[0,z_d]\right)}(D^*+x)$ for some $z \in B\big([-l^*,l^*]^{d-2}\times[l^*,2l^*]\times[s^*,2s^*]\big)$ with $z_d \in 2\hat{k}\mathbb{N}\Big\}$. Choose $v \in \{-1,1\}^{d-2}$ such that $x_i v_i \leq 0$ for all $i = 1,\dots,d-2$.[21] Define the following events:

$$A_x^1 := \left\{\omega \in \Omega : \begin{array}{l} \exists\, S_1 \subseteq \left(C^\beta_{\left(B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)+x\right)}(D^*+x) \cap \left(F^v_{(d-1)^+}(l^*,t^*)+x\right)\right) \text{ with } |S_1| \geq m^* \\ \text{such that for all distinct } y_1,y_2 \in S_1, L^\infty(y_1,y_2) \geq 4r^*+8\hat{k}+1 \end{array}\right\},$$

$$A_x^2 := \left\{\omega \in \Omega : \begin{array}{l} \exists\, y \in \left(C^\beta_{\left(B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)+x\right)}(D^*+x) \cap \left(F^v_{(d-1)^+}(l^*,t^*)+x\right)\right) \text{ such that} \\ \left(B([-r^*,r^*]^{d-2}\times[0,2r^*]\times[2r^*+2\hat{k},2r^*+4\hat{k}])+y\right) \subseteq C^0_{B\left([-r^*,r^*]^{d-2}\times[0,2r^*]\times[0,2r^*+4\hat{k}]\right)+y}(\{y\}) \end{array}\right\}.$$

By the construction of v, $\big(F^v_{(d-1)^+}(l^*,t^*)+x\big) \subseteq B\big([-l^*,l^*]^{d-2}\times l^*\times[0,t^*]\big)$. Since $2r^* \geq \max\{\beta_0,\dots,\beta_{\hat{k}-1}\}$, $t^* = 2r^*$, $l^* \geq r^*$, $s^* = t^*+2\hat{k}$, and $\big(F^v_{(d-1)^+}(l^*,t^*)+x\big) \subseteq B\big([-l^*,l^*]^{d-2}\times l^*\times[0,t^*]\big)$, it follows from

[21] Note that there may be more than one such v.

the construction of A_x^2 that $A_x^2 \subseteq A_x$. So, we have

$$\begin{aligned}
\mathbb{P}_p\Big(A_x\Big) \ge\ & \mathbb{P}_p\Big(A_x^2\Big) \\
\ge\ & \mathbb{P}_p\Big(A_x^2 \mid A_x^1\Big) \times \mathbb{P}_p\Big(A_x^1\Big) \\
\ge\ & \mathbb{P}_p\Big(A_x^2 \mid A_x^1\Big) \times \mathbb{P}_p\Bigg(A_x^1 \ \Bigg|\ \Big|\Big(C^{\beta}_{\left(B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)+x\right)}(D^*+x)\cap\big(F^v_{(d-1)^+}(l^*,t^*)+x\big)\Big)\Big| \ge n^*\Bigg) \\
& \times \mathbb{P}_p\Bigg(\Big|\Big(C^{\beta}_{\left(B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)+x\right)}(D^*+x)\cap\big(F^v_{(d-1)^+}(l^*,t^*)+x\big)\Big)\Big| \ge n^*\Bigg).
\end{aligned} \tag{A.33}$$

Define the mapping $S_1 : A_x^1 \longrightarrow \mathcal{P}(\mathbb{H})$ such that for all $\omega \in A_x^1$,

(i) $S_1(\omega) \subseteq \Big(C^{\beta}_{\left(B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)+x\right)}(D^*+x)\cap\big(F^v_{(d-1)^+}(l^*,t^*)+x\big)\Big)$, and

(ii) $|S_1(\omega)| \ge m^*$, where $L^\infty(y_1,y_2) \ge 4r^*+8\hat{k}+1$ for all distinct $y_1,y_2 \in S_1(\omega)$.

By the definition of m^*, A_x^1, S_1 and A_x^2, along with Remark A.1 and the construction of the β-backbend percolation process, we have

$$\begin{aligned}
& \mathbb{P}_p\Big(A_x^2 \mid A_x^1\Big) \\
\ge\ & \mathbb{P}_p\left(\begin{array}{l} \exists\, y \in S_1 \text{ such that} \\ \Big(B([-r^*,r^*]^{d-2}\times[0,2r^*]\times[2r^*+2\hat{k},2r^*+4\hat{k}])+y\Big) \subseteq C^{0}_{\left(B\left([-r^*,r^*]^{d-2}\times[0,2r^*]\times[0,2r^*+4\hat{k}]\right)+y\right)}(\{y\}) \end{array} \ \middle|\ A_x^1\right) \\
>\ & 1-\frac{\epsilon^*}{2}.
\end{aligned} \tag{A.34}$$

By the definition of A_x^1 and n^*,

$$\mathbb{P}_p\Bigg(A_x^1 \ \Bigg|\ \Big|\Big(C^{\beta}_{\left(B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)+x\right)}(D^*+x)\cap\big(F^v_{(d-1)^+}(l^*,t^*)+x\big)\Big)\Big| \ge n^*\Bigg) = 1, \tag{A.35}$$

and by Lemma A.6 and the construction of the β-backbend percolation process,

$$\mathbb{P}_p\Bigg(\Big|\Big(C^{\beta}_{\left(B\left([-l^*,l^*]^{d-1}\times[0,t^*]\right)+x\right)}(D^*+x)\cap\big(F^v_{(d-1)^+}(l^*,t^*)+x\big)\Big)\Big| \ge n^*\Bigg) > 1-\frac{\epsilon^*}{2}.$$

This, together with (A.33), (A.34), (A.35), yields

$$\mathbb{P}_p\Big(A_x\Big) > 1-\epsilon^*. \tag{A.36}$$

Note that the event A_x depends only on the edges in the set $\Big(B([-2l^*,2l^*]^{d-2}\times[-l^*,3l^*]\times[0,2s^*])\Big)^2$.

Since this set is finite, the proof of Lemma A.7 follows from (A.36) by using an argument similar to the one which we use to complete the proof of Lemma A.2 from (A.21). ■

Appendix B Proof of Proposition 3.2

Fix an arbitrary $l \in \mathbb{N}$. Using a similar argument as we have suggested for the proof of the "if" part of Theorem 3.4, we have $p_c^{\tilde{\beta}}(\mathbb{Q}_l^2) \leq p_c^{\beta}(\mathbb{Q}_l^2)$. We proceed to show that $p_c^{\beta}(\mathbb{Q}_{2l}^2) \leq p_c^{\tilde{\beta}}(\mathbb{Q}_l^2)$. In order to prove it, we show that for all $p \in [0,1]$, $\theta^{\tilde{\beta}}_{\mathbb{Q}_l^2}(p) > 0$ implies $\theta^{\beta}_{\mathbb{Q}_{2l}^2}(p) > 0$. If $p = 1$, then there is nothing to prove. Fix an arbitrary $p \in [0,1)$ such that $\theta^{\tilde{\beta}}_{\mathbb{Q}_l^2}(p) > 0$.

Since $\lim_{n\to\infty} \tilde{\beta}_{kn+i} = \beta_i$ for all $i \in \{0,\dots,k-1\}$, there exists $n^* \in \mathbb{N}$ such that $\tilde{\beta}_{kn+i} = \beta_i$ for all $n \geq n^*$ and all $i \in \{0,\dots,k-1\}$. Let $t^* \in k\mathbb{N}$ be such that $t^* \geq \max\{\beta_0,\dots,\beta_{k-1}\}$. Let us define the following events:

$$A := \left\{ \omega \in \Omega : \begin{array}{l} \text{there is an infinite open } \tilde{\beta}\text{-backbend path } (x^0, x^1, \dots) \text{ in } \mathbb{Q}_l^2 \\ \text{from the origin such that } x_d^j = kn^* + t^* \text{ for some } j \end{array} \right\}, \text{ and}$$

$$B := \left\{ \omega \in \Omega : \begin{array}{l} \text{there is an infinite open } \tilde{\beta}\text{-backbend path } (x^0, x^1, \dots) \text{ in } \mathbb{Q}_l^2 \\ \text{from the origin such that } x_d^j < kn^* + t^* \text{ for all } j \end{array} \right\}.$$

By the constructions of A and B, we have $\mathbb{P}_p(A \cup B) = \theta^{\tilde{\beta}}_{\mathbb{Q}_l^2}(p)$. Furthermore, since $B\big([-l,l]^{d-2} \times \mathbb{Z} \times [0, kn^* + t^*]\big)$ is a one-dimensional cylinder, the fact $p < 1$ implies that $\mathbb{P}_p(B) = 0$. Combining the facts that $\theta^{\tilde{\beta}}_{\mathbb{Q}_l^2}(p) > 0$, $\mathbb{P}_p(A \cup B) = \theta^{\tilde{\beta}}_{\mathbb{Q}_l^2}(p)$, and $\mathbb{P}_p(B) = 0$, we have $\mathbb{P}_p(A) > 0$. For all $s \in \mathbb{N}$, let us define the following event:

$$A_s := \left\{ \omega \in \Omega : \begin{array}{l} \text{there is an infinite open } \tilde{\beta}\text{-backbend path } \pi = (x^0, x^1, \dots) \text{ in } \mathbb{Q}_l^2 \\ \text{from the origin such that } |x_{d-1}^j| < s \text{ for all } j \text{ with } h^j(\pi) < kn^* + t^* \end{array} \right\}.$$

By the constructions of A and A_s, it follows that $A_s \underset{s\to\infty}{\uparrow} A$. Since $\mathbb{P}_p(A) > 0$, this implies that there exists $s^* \in \mathbb{N}$ such that

$$\mathbb{P}_p(A_{s^*}) > 0. \tag{B.1}$$

Claim B.1. $A_{s^*} \subseteq \left\{ \omega \in \Omega : \left| C^{\beta}_{B\left([-l,l]^{d-2}\times\mathbb{Z}\times[kn^*,\infty]\right)}\left(B\big([-l,l]^{d-2} \times [-s^*, s^*] \times kn^*\big)\right)\right| = \infty \right\}.$

Proof of Claim B.1. Fix an arbitrary configuration $\omega^* \in A_{s^*}$. Let $\pi = (x^0, x^1, \dots)$ be an infinite open $\tilde{\beta}$-backbend path in $\mathbb{Q}_l^2$ from the origin such that $|x_{d-1}^j| < s^*$ for all j with $h^j(\pi) < kn^* + t^*$. Since

$B([-l,l]^{d-2}\times[-s^*,s^*]\times[0,kn^*+t^*])$ is a finite set of vertices, it follows from the assumptions on π that there exists a finite sub-path $\pi_1=(x^0,\ldots,x^{m^*})$ of π from the origin such that π_1 is in $B([-l,l]^{d-2}\times[-s^*,s^*]\times[0,kn^*+t^*])$ and $x_d^{m^*}=kn^*+t^*$. Let z^* be the last vertex of π_1 such that $z_d^*=kn^*$. Note that by construction, $z^*\in B([-l,l]^{d-2}\times[-s^*,s^*]\times kn^*)$. Consider the sub-path π^* of π from z^*. Clearly, π^* is an infinite open path.

First, we show that π^* is in $B([-l,l]^{d-2}\times\mathbb{Z}\times[kn^*,\infty])$. We divide π^* into two parts. The first part π_1^* is the sub-path of π^* from z^* to x^{m^*}, and the second part π_2^* is the sub-path of π^* from x^{m^*}. By construction, π_1^* is in $B([-l,l]^{d-2}\times[-s^*,s^*]\times[kn^*,kn^*+t^*])$. Furthermore, because of the facts that π is a $\tilde{\beta}$-backbend path in $\mathbb{Q}_l^2$, $x_d^{m^*}=kn^*+t^*$, $\tilde{\beta}_i=\beta_i$ for all $i\geq kn^*$, and $t^*\geq\max\{\beta_0,\ldots,\beta_{k-1}\}$, it follows by construction that π_2^* is in $B([-l,l]^{d-2}\times\mathbb{Z}\times[kn^*,\infty])$. Since π_1^* is in $B([-l,l]^{d-2}\times[-s^*,s^*]\times[kn^*,kn^*+t^*])$, and π_2^* is in $B([-l,l]^{d-2}\times\mathbb{Z}\times[kn^*,\infty])$, it follows that π^* is in $B([-l,l]^{d-2}\times\mathbb{Z}\times[kn^*,\infty])$.

In order to complete the proof of Claim B.1, it remains to show that π^* is a β-backbend path. Assume for contradiction that π^* is not a β-backbend path. Since π^* is not a β-backbend path, there exists a vertex x^* in π^* such that $x_d^*<h^*-\beta_{h^*}$, where h^* is the record level attained by the path π^* till x^*. Let $\hat{h}$ be the record level attained by the path π till x^*. Clearly, $\hat{h}\geq h^*\geq kn^*$. Because π is a $\tilde{\beta}$-backbend path, it must be that $x_d^*\geq\hat{h}-\tilde{\beta}_{\hat{h}}$. Since $\hat{h}\geq kn^*$ and $\tilde{\beta}_i=\beta_i$ for all $i\geq kn^*$, this implies $x_d^*\geq\hat{h}-\beta_{\hat{h}}$. Combining the facts that $x_d^*<h^*-\beta_{h^*}$ and $x_d^*\geq\hat{h}-\beta_{\hat{h}}$, we have

$$h^*-\beta_{h^*}>\hat{h}-\beta_{\hat{h}}. \tag{B.2}$$

The assumptions on β imply that $\beta_l-\beta_m\leq l-m$ for all $l\geq m$. Since $\hat{h}\geq h^*$, this yields $\beta_{\hat{h}}-\beta_{h^*}\leq\hat{h}-h^*$, a contradiction to (B.2). So, it must be that π^* is a β-backbend path. This completes the proof of Claim B.1. □

Now, we complete the proof of Proposition 3.2. Because $B([-l,l]^{d-2}\times[-s^*,s^*]\times kn^*)$ is a finite set, (B.1) and Claim B.1 together imply that there exists some $x\in B([-l,l]^{d-2}\times[-s^*,s^*]\times kn^*)$ such that $\mathbb{P}_p\left(|C^{\beta}_{(x+\mathbb{Q}^2_{2l})}(\{x\})|=\infty\right)>0$. Furthermore, since β is k-cyclic and $x_d=kn^*$, by the construction of the β-backbend percolation process, we have $\mathbb{P}_p\left(|C^{\beta}_{\mathbb{Q}^2_{2l}}|=\infty\right)=\mathbb{P}_p\left(|C^{\beta}_{(x+\mathbb{Q}^2_{2l})}(\{x\})|=\infty\right)$, and hence $\mathbb{P}_p\left(|C^{\beta}_{\mathbb{Q}^2_{2l}}|=\infty\right)>0$. This completes the proof of Proposition 3.2. ■

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