Stability in Matching with Couples having Responsive Preferences[*]

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Abstract

This paper studies matching markets where institutions are matched with possibly more than one individual. The matching market contains some couples who view the pair of jobs as complements. First, we show by means of an example that a stable matching may fail to exist even when both couples and institutions have responsive preferences. Next, we provide conditions on couples' preferences that are necessary and sufficient to ensure a stable matching for every preference profile where institutions may have any responsive preference. Finally, we do the same with respect to institutions' preferences, that is, we provide conditions on institutions' preferences that are necessary and sufficient to ensure a stable matching for every preference profile where couples may have any responsive preference.

1 Introduction

1.1 Background of the problem

A huge literature has been developed on various market designs for finding an "optimal" matching procedure in a labour market. In many centralized labour markets, stability is an important condition for optimality. A matching is stable when there do not exist institution-individual pairs that are not matched to each other, but when being matched are better off compared to their present allocation. Such pairs are also known as blocking pairs. For labour markets consisting of institutions (referred to here as hospitals) and individuals (here, called doctors), stable matchings do exist under appropriate domain restrictions.

Roth[12] was the first to point out that the presence of couples in labour markets may exclude the existence of a stable matching. This can happen because couples may view pairs of jobs as complements and therewith the assumption on the independence of the individual choices might not apply.

However, Kelso and Crawford[8], Roth[13], Alkan and Gale[2] and Hatfield and Milgrom[7] show that a sufficient amount of substitutability can guarantee the existence of a stable matching. These papers assume substitutability of preferences of the hospitals over sets of doctors. Later, Klaus and Klijn[9] assume responsiveness of couples' preferences over ordered pairs of hospitals and show that the existence of a stable matching is guaranteed in such settings. Responsiveness means that unilateral improvements according to the preference of one couple member are beneficial for the couple.

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1.2 Our motivation

As mentioned, we assume that the institutions are hospitals and the individuals are doctors, where the subset of doctors includes couples. We assume that couples' preferences are responsive over the pair of hospitals and hospitals' preferences are responsive over the set of doctors.

It is interesting to note that given the individual preferences of the couple members, more than one (joint) couple preference satisfies responsiveness. For example, suppose we have a couple $c = \{f, m\}$ and two hospitals h_1 and h_2 such that f prefers h_1 to h_2 but m prefers h_2 to h_1. Suppose further that (x, y) denotes an allocation of this couple where f is matched with x and m is matched with y. Then we know by responsiveness that (h_1, h_2) is preferred to both (h_1, h_1) and (h_2, h_2) whereas, (h_1, h_1) and (h_2, h_2) are both preferred to (h_2, h_1). However, responsiveness does not give us an ordering between the pairs (h_1, h_1) and (h_2, h_2). Thus there are different complete responsive couple preference extensions. Similarly, given a preference of a hospital over doctors, we can have more than one responsive preference over subsets of doctors.

Allowing for complete responsive preferences for both couples of doctors and hospitals is not innocuous, as in such situations stable matchings may fail to exist. On the other hand, allowing that the pairs (h_1, h_1) and (h_2, h_2) are comparable in the above mentioned example of couple preference seems to be natural. Similarly, it seems natural that hospitals admitting teams of doctors compare, for instance, two pairs of doctors, where in one pair, there is one A ranked doctor and one C ranked doctor, and in the other pair there are two B ranked doctors. As we will see, allowing for such comparisons does affect the stability of matchings. However, it is worthwhile to mention that such comparisons might not always be possible even when the preferences of hospitals satisfy responsiveness. For example, by a standard replication of hospitals up to their capacities to create a one-to-one matching, these pairs of doctors are actually considered incomparable. This is because, for the hospital the B ranked doctor is better than the C ranked doctor but is

worse than the A ranked doctor. Thus, replication implies that these two sets of doctors are incomparable. The possibility of allowing for all such above mentioned comparisons is the main reason why results presented here differ from Klaus and Klijn[9], who found stable matchings in every case of couples' responsive preferences.

1.3   Our contribution

In this paper, we formalize our idea of complete responsive preferences of hospitals and couples and study its consequences for the existence of a stable matching.

First, we show by means of an example that stable matchings may not exist at arbitrary responsive preference profiles. Then, we provide a condition on couples' preferences that is necessary and sufficient for guaranteeing the existence of a stable matching for every responsive extension of hospitals' preferences over sets of doctors. We call this condition *extreme-altruism*. We explain this condition by means of the following example. Suppose that there are k hospitals $\{h_1, \ldots, h_k\}$ and a couple. Suppose further that h_i is preferred to h_j for all $1 \leq i < j \leq k$ by both members of the couple. Then, extreme-altruism says that, for any $1 \leq i < j \leq k$, the couple prefers the allocation where one member is matched with h_i and the other is matched with h_k compared to the allocation where both of them are matched with h_j.

Next, we provide conditions on hospitals' preferences that are necessary and sufficient for guaranteeing the existence of a stable matching for every responsive extension of couples' preferences. We call this condition *aversion to couple diversity*. We explain this condition by means of the following example. Consider a hospital h with preference P_h over individual doctors. Let $c = \{f, m\}$ be a couple and let d, d' be two other doctors (possibly another couple). Suppose that $f P_h d P_h d' P_h m$. As we have discussed earlier, responsiveness does not put any restriction on h's preference over the relative ordering of the sets of doctors $\{f, m\}$ and $\{d, d'\}$. A hospital is averse of diverse couples if in such situations it prefers $\{d, d'\}$ to $\{f, m\}$. Loosely put,

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this property asserts that a hospital does not like to employ couples whose members are relatively more dissimilar (with respect to its preference over individual doctors).

Another important contribution of our paper is that, not only we characterize profiles where stable matchings exist, but we also provide algorithms that produce a stable matching whenever that exists.

Thus, we feel that our paper provides a more complete picture on the issue of the existence of stable matchings when hospitals and couples have responsive preferences, and consequently complements the work of Klaus and Klijn[9].

1.4   Organization of the paper

In the next section, we lay out the framework of our model and formally provide all the necessary definitions. We also state and describe an algorithm for matching doctors with hospitals, which we use throughout this paper. In Section 3, we provide an example showing that the existence of a stable matching is not guaranteed at arbitrary responsive preferences profiles. In Section 4, we present conditions on couples' preferences that are necessary and sufficient for guaranteeing the existence of a stable matching for arbitrary responsive extensions hospitals' preferences, and in Section 5, we present conditions on hospitals' preferences that are necessary and sufficient for guaranteeing the existence of a stable matching for arbitrary responsive extensions of couples' preferences. We conclude with Section 6 where we explain the difference between our paper and Klaus and Klijn[9] by means of a formal example.

2   The framework

We consider many-to-one matching between doctors and hospitals. We denote by H a finite set of hospitals. Each hospital $h \in H$ has a finite capacity, denoted by κ_h.

We denote by D a finite set of doctors. We assume that $D = F \cup M \cup S$ where F, M, S are pairwise disjoint sets with $|F| = |M|$. Doctors in F and M are respectively denoted by $f_1, f_2, \ldots$ and $m_1, m_2, \ldots$. Each pair of doctors (f_i, m_i) forms a couple. We denote the set of couples by $C = \{\{f_1, m_1\}, \{f_2, m_2\}, \ldots\}$ and a generic couple by $c = \{f, m\}$. The doctors in S are not part of any couple. We use the notation λ, $\lambda \notin H$, to denote a dummy hospital. The role of dummy hospital is that whenever a doctor is matched with a dummy hospital it means that she/he is unemployed.

Throughout this paper, we assume $|H| \geq 2$, $|D| \geq 4$, $|C| \geq 1$, and $\kappa_h \geq 2$ for all $h \in H$ and $\kappa_\lambda = |D|$.

For notational convenience, whenever it is clear from the context, we do not always use braces for denoting singleton hospitals, doctors or couples.

2.1 Matching

DEFINITION 1 *A matching μ is a correspondence from $H \cup \{\lambda\}$ to D such that $\mu(h_1)$ and $\mu(h_2)$ are disjoint for all distinct $h_1, h_2 \in H$, and $|\mu(h)| \leq \kappa_h$ for all $h \in H$.*

For ease of presentation, whenever $d \in \mu(h)$ for some $d \in D$ and $h \in H$, we write $\mu(d) = h$. As we have mentioned, $\mu(d) = \lambda$ means d is unemployed. For a couple $c = \{f, m\} \in C$ and for hospitals $h, h' \in H \cup \{\lambda\}$, we write $\mu(c) = (h, h')$ to mean $\mu(f) = h$ and $\mu(m) = h'$. Further, for a hospital $h \in H$ and a matching μ, the number $\kappa_h - |\mu(h)|$ is referred to as the number of vacant positions in h at μ.

2.2 Preferences

In this section, we introduce the notion of preferences of doctors and hospitals, and present some restrictions on them. For a set X, we denote by $\mathbb{L}(X)$ the set of linear orders on X, i.e., complete, reflexive, transitive, and antisymmetric binary relations over X. An element of $\mathbb{L}(X)$ is called a preference (over X). For any $i \in H \cup D \cup C$, R_i denotes a preference of i and

P_i denotes its strict part. Since a preference is antisymmetric xR_iy implies either $x = y$ or xP_iy. We say x is weakly preferred to y to mean xR_iy, and x is (strictly) preferred to y to mean xP_iy. For $P_i \in \mathbb{L}(X)$ and $k \le |X|$, we define the k-th ranked element in P_i, denoted by $r_k(P_i)$, as the element x of X such that $|\{y \in X : yR_ix\}| = k$.

2.2.1 Preferences of hospitals

Each hospital $h \in H$ has a set D_h of acceptable doctors. The implication of the set of acceptable doctors is that a hospital prefers to have any doctor from the set to having a vacant spot and prefers to have a vacant spot to having a doctor outside the set. A preference of hospital h, denoted by $\bar{P}_h$, is a linear order over D_h, that is, $\bar{P}_h \in \mathbb{L}(D_h)$. We assume that the dummy hospital λ finds all doctors acceptable. Thus, $D_\lambda = D$. Also, λ is indifferent between all doctors.

For any hospital $h \in H$, a preference $\bar{P}_h$ over individual doctors is extended to a preference P_h over feasible subsets of acceptable doctors $\{D' \subseteq D_h : |D'| \le \kappa_h\}$ in a responsive manner.

DEFINITION 2 *A preference $P_h \in \mathbb{L}(\{D' \subseteq D_h : |D'| \le \kappa_h\})$ is* responsive *if*

  (i) *for all $D' \subseteq D_h$ with $D' \ne \emptyset$ and $|D'| \le \kappa_h$, $D'P_h\emptyset$.*

  (ii) *for all $d, d' \in D_h$, $\{d\}R_h\{d'\}$ if and only if $d\bar{R}_hd'$, and*

  (iii) *for all $D', D'' \subseteq D_h$ with $|D'| < \kappa_h$, $|D''| < \kappa_h$ and all $d \in D_h \backslash (D' \cup D'')$, $(D' \cup \{d\})P_h(D'' \cup \{d\})$ if and only if $D'P_hD''$.*

We extend P_h over the set of all feasible (including the unacceptable ones) subsets of doctors $\{D' \subseteq D : |D'| \le \kappa_h\}$ in a natural way: for all $D' \subseteq D$ such that $|D'| \le \kappa_h$ and D' is not a subset of D_h, we have $\emptyset P_hD'$.

For notational convenience, for any hospital $h \in H$, any couple $c = \{f, m\} \in C$, and any doctor $d \in D \setminus \{f, m\}$, we write dP_hc to mean dP_hf and dP_hm.

2.2.2 Preferences of doctors and couples

A preference of a doctor $d \in D$, denoted by P_d, is an element of $\mathbb{L}(H \cup \{\lambda\})$. A preference of a couple $c = \{f, m\} \in C$, denoted by P_c, is an element of $\mathbb{L}((H \cup \{\lambda\})^2)$. We call a preference of a couple responsive if a unilateral improvement in the preference of one member of the couple is beneficial for the couple.

DEFINITION **3** *Let $c = \{f, m\} \in C$ be a couple. Let P_f be a preference of f and P_m be a preference of m. A preference $P_c \in \mathbb{L}((H \cup \{\lambda\})^2)$ of the couple c is called responsive (with respect to P_f and P_m) if for all $h, h_1, h_2 \in H \cup \{\lambda\}$, we have*

(i) *$(h_1, h)P_c(h_2, h)$ if and only if $h_1 P_f h_2$, and*

(ii) *$(h, h_1)P_c(h, h_2)$ if and only if $h_1 P_m h_2$.*

It is worth noting that for given P_f and P_m, there can be several responsive preferences of the couple $c = \{f, m\}$. However, for a given responsive preference P_c of the couple $c = \{f, m\}$, the "marginal preferences" P_f and P_m for f and m, respectively, are unique.

2.2.3 Preference profiles and matching problems

A preference profile is a collection of responsive preferences for all hospitals in H, all doctors in D, and all couples in C. Formally, a preference profile P is a tuple of preferences $(\{P_d\}_{d \in D}, \{P_c\}_{c \in C}, \{P_h\}_{h \in H})$, where for all $d \in D$, $c \in C$ and $h \in H$, P_d is a preference of doctor d, P_c is a responsive preference of couple c, and P_h is a responsive preference of hospital h over acceptable and feasible sets of doctors, respectively. Note that for any hospital $h \in H$, D_h is inherent in P_h, that is, P_h automatically specifies the acceptable set D_h of the hospital.

A matching problem is a tuple consisting of a set of hospitals with corresponding capacities, a set of doctors with its partition into F, M, S, and a preference profile.

2.3 Stability

There are different notions of stability based on different types of permissible blocking coalitions.

Let μ be a matching and P be a preference profile. We say a hospital h *prefers* to have a set of doctors D' (possibly empty) to a subset of doctors in $\mu(h)$ if there is $D'' \subseteq \mu(h)$ with $D' \cap D'' = \emptyset$ such that $\{(\mu(h) \setminus D'') \cup D'\}P_h\mu(h)$. In other words, a hospital prefers to have a set of doctors D' if it is possible for the hospital to be better off by removing some existing/matched doctors D'' and taking the doctors in D'. Similarly, we say a doctor d (or a couple c) *prefers* a hospital h to $\mu(d)$ (or a pair of hospitals (h, h') to $\mu(c)$) if $hP_d\mu(h)$ (or $(h, h')P_c\mu(c)$). Note that if a hospital prefers a set of doctors to its assignment at μ, then by definition, that hospital is not matched with any of those doctors at μ. Moreover, it could also be that h has some unacceptable doctors D'' in $\mu(h)$, thus h prefers $D' = \emptyset$ to $\mu(h)$. Similarly, if a doctor (or a couple) prefers a hospital (or a pair of hospitals) to its assignment at μ, then that doctor (or at least one member of that couple) is not matched with the hospital (or the corresponding hospital) at μ.

Now, we define the notion of blocking. Note that, since λ is indifferent between all sets of doctors, and $D_\lambda = D$, thus λ always prefers to have any doctor than not having that doctor. First, we introduce the notion of blocking between a hospital and a doctor in S.

DEFINITION 4 *Let $s \in S$, $h \in H \cup \{\lambda\}$ and let μ be a matching. Then (h, s) blocks μ if h prefers s to $\mu(h)$ and s prefers h to $\mu(s)$.*

Next, we define the notion of blocking between a pair of hospitals and a couple.

DEFINITION 5 *Let μ be a matching and let $c = \{f, m\} \in C$ and $(h_f, h_m) \in (H \cup \{\lambda\})^2$. Then, $((h_f, h_m), c)$ blocks μ if c prefers (h_f, h_m) to $\mu(c)$ and*

(i) if $h_f \neq h_m$ and $\mu(f) \neq h_f$, then h_f prefers f to $\mu(h_f)$,

(ii) if $h_f \neq h_m$ and $\mu(m) \neq h_m$, then h_m prefers m to $\mu(h_m)$, and

(iii) if $h_f = h_m$, then h_f prefers $\{f, m\}$ to $\mu(h_f)$.

It is worth mentioning that the blocking notion takes complementarity of a couple being accepted into account (by allowing the notion of a hospital being interested in a couple) but it does not take the couple into account when accepting single doctors and possibly removing members of a couple. In other words, there is an asymmetry here.

We consider this asymmetry in our model since it is not practical for big institutions like hospitals to consider the possibility of losing a member of a couple while removing the other member. This is because this possibility depends on factors like which hospital the removed member will join, whether the couple prefers to be together in that hospital, etc. Clearly, such situations can only be modeled by using a farsighted notion of blocking, which would complicate the model considerably.

Thus, by allowing the notion of a hospital being interested in a couple, the blocking definition takes complementarity of a couple being accepted into account. However, the hospital does not take this into account while accepting single doctors at the cost of removing a member of the couple from the hospital. The asymmetry arising here is the main reason, why the results obtained in this chapter are different to the results obtained by choice function approach in many-to-many matchings. [1]

Whenever a matching μ is blocked by $((h_f, \mu(m)), c)$ for some $c = \{f, m\} \in C$ and some $h_f \in H \cup \{\lambda\}$, for ease of presentation we say that μ is blocked by (h_f, f). Similarly we say that μ is blocked by (h_m, m) if μ is blocked by $((\mu(f), h_m), c)$.

Our next remark follows from the responsiveness of couples' preferences and Definition 5.

REMARK **1** *Let μ be a matching and $c = \{f, m\}$ be a couple. Suppose for some $x \in \{f, m\}$ and some hospital $h_x \in H$, we have $h_x P_x \mu(x)$ and $((\mu(h_x) \setminus d) \cup \{x\}) P_{h_x} \mu(h_x)$ for some $d \in \mu(h_x) \setminus \{f, m\}$, then (h_x, x) blocks μ.*

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[1]See Konishi and Ünver[11], Echenique and Oviedo[3], Hatfield and Kojima[6], for notions of stability in many-to-many matchings.

A matching is stable if it cannot be blocked by any blocking pair. More formally, we get the following definition.

DEFINITION **6** *A matching μ is stable, if*

(i) *for all $h \in H \cup \lambda$ and $s \in S$, (h, s) does not block μ,*

(ii) *for all $(h_f, h_m) \in (H \cup \lambda)^2$ and $c \in C$, $((h_f, h_m), c)$ does not block μ, and*

(iii) *for all $h \in H$, $(h, \emptyset)$ does not block $\mu(h)$, i.e., h does not prefer $\emptyset$ to $\mu(h)$.*

Now, we define the concept of individual rationality.

DEFINITION **7** *A matching μ is individually rational if*

(i) *for all $s \in S$, $\mu(s) R_s \lambda$,*

(ii) *for all $c \in C$, $\mu(c) R_c(\lambda, \lambda)$, and*

(iii) *for all $h \in H$ and all $d \in \mu(h)$, $d \in D_h$.*

The next remark follows from the definition of stability.

REMARK **2** *Every stable matching is individually rational.*

2.4 Algorithm

In this section we present a well-known algorithm called doctor proposing deferred acceptance algorithm (DPDA). This algorithm was introduced by Gale and Shapley[4]. [2] Our proofs for the existence of stable matchings use a modification of DPDA. In what follows, we give a very short description of this algorithm. Take a profile P. Then, the DPDA algorithm at P goes as follows.

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[2]See Knuth[10], Gusfield and Irving[5], Roth and Sotomayer[14], Aldershof and Carducci[1] for additional results on stable matching problem in two sided matching.

DPDA: In step 1 of the algorithm, all doctors simultaneously propose to their most preferred hospitals. Each hospital $h \in H$ provisionally accepts the most preferred doctors according to P_h. If a hospital receives more than κ_h proposals, then it rejects all the doctors which do not belong to its κ_h most preferred doctors. In any step k, the unmatched doctors propose to their most preferred hospital from the remaining set of hospitals who have not rejected them in any of the earlier steps. In any step of DPDA, since any hospital $h \in H$ accepts the most preferred collection of doctors according to P_h, it may reject some doctors that it had provisionally accepted earlier. Hospitals whose provisional list of accepted doctors is less than their maximum capacity can still add to their accepted list if they receive fresh proposals. Thus the algorithm terminates when each doctor is matched with some hospital or has been rejected by all acceptable hospitals.

REMARK **3** *Note that in DPDA, each individual doctor proposes according to his/her individual preference. Thus, couples' preferences do not play any role in this algorithm.*

It is well-known that the outcome of DPDA is optimal for doctors. That is, some doctor is worse off at every other stable matching. Moreover, by responsiveness and the structure of DPDA, it follows that the outcome of DPDA is individually rational.

The following remark follows directly from the definition of DPDA.

REMARK **4** *Let μ be the outcome of DPDA. Let $d \notin \mu(h)$ for some $d \in D$ and $h \in H$. Then, $hP_d\mu(d)$ implies $d'P_hd$ for all $d' \in \mu(h)$.*

In the following lemmas, we show that the outcome of DPDA cannot be blocked by a hospital and a single doctor or by a pair of different hospitals and a couple. Some of these results are well known outcomes of DPDA, but we prove them nevertheless for the sake of completeness.

LEMMA **1** *The outcome of DPDA cannot be blocked by a pair (h, s) for any $h \in H$ and any $s \in S$.*

Proof: Let μ be the outcome of DPDA. Assume for contradiction that (h, s) blocks μ for some $h \in H$ and some $s \in S$. Since the outcome of DPDA is individually rational, $sR_h\lambda$. Since $hP_s\mu(s)$, by the definition of DPDA and Remark 4, either s has not proposed to h during the DPDA or all the doctors in $\mu(h)$ are preferred to s according to P_h. If s has not proposed to h during DPDA, then we have $\mu(s)P_s h$, a contradiction to the fact that (h, s) blocks μ. So, suppose $dP_h s$ for all $d \in \mu(h)$. Then, by responsiveness of hospitals' preferences, we have $\mu(h)P_h((\mu(h) \setminus \{d\}) \cup s)$ for all $d \in \mu(h)$, and consequently, hospital h will not block with s. This completes the proof of the lemma. $\square$

LEMMA **2** *The outcome of DPDA cannot be blocked by $((h_1, h_2), c)$ for any $h_1, h_2 \in H$ such that $h_1 \neq h_2$ and for any $c \in C$.*

Proof: Let μ be the outcome of DPDA. Assume for contradiction that μ is blocked by $((h_1, h_2), c)$. Let $\mu(f) = h_f$ and $\mu(m) = h_m$. By the definition of a block, $(h_1, h_2)P_c(h_f, h_m)$.

Suppose $h_f R_f h_1$ and $h_m R_m h_2$. Since $(h_1, h_2) \neq (h_f, h_m)$, this means $(h_f, h_m)P_c(h_1, h_2)$, a contradiction. Now, suppose $h_1 P_f h_f$ or $h_2 P_m h_m$. Without loss of generality, assume $h_2 P_m h_m$. Since the outcome of DPDA is individually rational, $h_2 \neq \lambda$. Because $h_2 P_m h_m$, by Remark 4, m proposed to h_2 at some step of DPDA and got rejected. Since $h_1 \neq h_2$, by Lemma 1, we have $\mu(h_2)P_{h_2}((\mu(h_2) \setminus \{d\}) \cup m)$ for all $d \in \mu(h_2)$. However, this contradicts the definition of a block. $\square$

In what follows, we give a lemma which shows that the outcome of DPDA cannot be blocked by a pair of dummy hospitals and a couple.

LEMMA **3** *The outcome of DPDA cannot be blocked by $((\lambda, \lambda), c)$ for any $c \in C$.*

Proof: Let μ be the outcome of DPDA. Assume for contradiction that μ is blocked by $((\lambda, \lambda), c)$. Let $\mu(f) = h_f$ and $\mu(m) = h_m$. By the definition

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of block, $(\lambda, \lambda)P_c(h_f, h_m)$. By responsiveness, this means $\lambda P_x h_x$ for some $x \in \{f, m\}$. However, by the definition of DPDA, $\lambda P_x h_x$ means x proposed to λ before proposing to h_x and got rejected. This is contradiction as, by our assumption, λ cannot reject a doctor. $\square$

In the following lemmas, we give conditions when an outcome of DPDA cannot be blocked by a pair of same hospitals and a couple.

LEMMA 4 *Suppose μ is an outcome of DPDA. Then for any $h \in H$ and any $c = \{f, m\} \in C$,*

(i) *$fP_h m$ and $hP_f \mu(f)$ implies $((h, h), c)$ cannot block μ, and*

(ii) *$mP_h f$ and $hP_m \mu(m)$ implies $((h, h), c)$ cannot block μ.*

Proof: Assume without loss of generality that $fP_h m$ and $hP_f \mu(f)$. By the definition of DPDA, f proposed to h and got rejected. By Remark 4, $dP_h f$ for all $d \in \mu(h)$. Because $fP_h m$, $\{d, d'\}P_h\{f, m\}$ for all $d, d' \in \mu(h)$. This means h cannot block μ with c. This completes the proof. $\square$

LEMMA 5 *Suppose μ is an outcome of DPDA. Then for any $h \in H$ and any $c = \{f, m\} \in C$, $((h, h), c)$ cannot block μ if $h = \mu(x)$ for some $x \in \{f, m\}$*

Proof: Without loss of generality, let $h = \mu(f)$. Let $((h, h), c)$ block μ.

By the definition of a block, $(h, h)P_c(\mu(f), \mu(m))$. Thus, by responsiveness of couples' preferences, $hP_m^0 h_m$. By the definition of DPDA, this means m proposed to h and got rejected. Since $f \in \mu(h)$, by Lemma 1 and the definition of block, we have $\mu(h)P_h((\mu(h) \setminus \{d\}) \cup m)$ for all $d \in \mu(h)$. This means h will not block μ with c, which is a contradiction. $\square$

REMARK 5 *If a doctor $d \in D$ and hospital $h \in H$ are each other's top ranked alternative, then trivially for a stable match, they must be matched to each other.*

3 Stable matching is not guaranteed at arbitrary responsive profiles

In this section, we show by means of two examples that existence of a stable matching is not guaranteed at arbitrary responsive preference profiles. The two examples are slightly different to suit the subsequent sections.

EXAMPLE 1 Let $H = \{h_1, h_2, h_3\}$, $\kappa_h = 2$ for all $h \in \{h_1, h_2, h_3\}$, and $D = \{d_1, d_2, d_3, d_4, f, m\}$ where $c = \{f, m\}$ is a couple.

Suppose $r_1(P_{d_1}) = h_2$ and $r_2(P_{d_1}) = r_1(P_{d_2}) = h_1$. Further, $r_1(P_{d_3}) = h_2$ and $r_1(P_{d_4}) = h_3$. For the couple, suppose $h_2 P_f h_1$, $h_2 P_f h_3$ and $h_1 P_m h_3$ but $(h_1, h_1) P_c (h_2, h_3)$. Finally $h P_x \lambda$ for all $x \in \{f, m\}$ and all $h \in H$.

The above mentioned preferences along with the preferences of all the hospitals over individuals are given in Table 1. The preference of all hospitals over pairs of doctors can be any responsive preference over pairs of doctors. However $\{f, m\} P_{h_1} \{d_1, d_2\}$.

The couple's preference over pairs of hospitals, where one member is matched and the other one is unmatched are assumed to be responsive and ranked below the pairs of hospitals.

P_{d_1}	P_{d_2}	P_{d_3}	P_{d_4}	P_f	P_m	P_c	$\bar{P}_{h_1}$	$\bar{P}_{h_2}$	$\bar{P}_{h_3}$
h_2	h_1	h_2	h_3	⋮	⋮	⋮	d_3	d_3	d_4
h_1	⋮	⋮	⋮	h_2	h_1	(h_1, h_1)	d_4	d_4	d_3
⋮				⋮	⋮	⋮	f	f	d_1
				h_1	h_3	(h_2, h_3)	d_1	d_1	f
				⋮	⋮	⋮	d_2	m	m
							m	d_2	d_2

Table 1

The couple and h_1 have responsive preferences. In what follows, we argue that there is no stable matching for the preference profile given in Table 1.

Assume for contradiction, that there exists a stable matching μ for the given preference profile. By Remark 5, it must be that $\mu(d_3) = h_2$ and $\mu(d_4) = h_3$, as the doctors and hospitals are each other's top ranked alternative. Thus, there is potentially only one vacancy to be filled in h_2 and h_3 respectively.

Particularly, it is not possible for the couple to be matched to (h_2, h_2) and (h_3, h_3) for a stable matching. Also, since the couple prefers to be matched with any two hospitals than having at least one member of the couple unmatched and we have sufficiently many number of vacancies in all the three hospitals, thus, $\{\mu(f), \mu(m)\} \subseteq \{h_1, h_2, h_3\}$.

Now we look at the following allocations of the couple in μ.

(i) Suppose $\mu(c) = (h_2, h_1)$.

Note that $f P_{h_2} d_1$. Since $d_i \bar{P}_{h_1} m$ for $i \in \{1, 2\}$, $r_2(P_{d_1}) = r_1(P_{d_2}) = h_1$. So, $\mu(h_2) = \{d_3, f\}$. As $\kappa_{h_1} = 2$, therefore $d_i \notin \mu(h_1)$ for some i. Thus, stability of μ implies that (h_1, d_i) blocks μ.

(ii) Suppose $\mu(c) = (h_1, h_1)$.

Then (h_1, d_2) blocks μ.

(iii) Suppose $\mu(c) = (h_2, h_3)$.

Since $\{f, m\} P_{h_1} \{d_1, d_2\}$ and $(h_1, h_1) P_c (h_2, h_3)$, $((h_1, h_1), c)$ blocks μ.

(iv) Suppose $\mu(c) = (h_1, h_3)$.

Since $h_2 P_f h_1$, responsiveness implies $(h_2, h_3) P_c (h_1, h_3)$. This together with the fact that $f \bar{P}_{h_2} d_1 \bar{P}_{h_2} d_2$ implies $((h_2, h_3), c)$ blocks μ.

(v) Suppose $\mu(c) = (h_3, h_1)$.

Since $h_2 P_f h_3$, responsiveness implies $(h_2, h_1) P_c (h_3, h_1)$. This together with the fact that $f \bar{P}_{h_2} d_1 \bar{P}_{h_2} d_2$ implies $((h_3, h_1), c)$ blocks μ.

(vi) Suppose $\mu(c) = (h, h_2)$ for some $h \in \{h_1, h_3\}$.

Thus, $d_1 \notin \mu(h_2)$. Since $r_1(P_{d_1}) = h_2$ and $d_1 P_{h_2} m$, stability of μ implies that (h_2, d_1) blocks μ.

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Cases (i)-(vi) together are exhaustive. Thus, it follows that there is no stable matching for the given preference profile in Table 1.

Note that, the arguments of this example can also be used to give an example with no stable matching when we consider h_2 and h_3 as the same hospital. Thus, we just consider two vacancies for both the hospitals while excluding d_3 and d_4 from the set of doctors. It follows from the above argument that there does not exist a stable matching when $(h_1, h_1)P_c(h_2, h_2)$ with $h_2 P_f h_1$, $h_1 P_m h_2$ and the preferences of hospitals over $\{f, m, d_1, d_2\}$ as given by $\bar{P}_{h_1}$ and $\bar{P}_{h_2}$.

EXAMPLE **2** Let $H = \{h_1\}$, $\kappa_{h_1} = 2$ and $D = \{d_1, d_2, f, m\}$ where $c = \{f, m\}$ is a couple.

The preference of the hospital over individuals and pairs of individuals, preference of individual doctors and the couple preferences are given in Table 2.

The preference of h_1 over the pairs of doctors where at least one position is vacant is assumed to ranked below the shown pairs.

$\bar{P}_{h_1}$	P_{d_1}	P_{d_2}	P_f	P_m	P_c	P_{h_1}
f	h_1	h_1	λ	h_1	(λ, h_1)	$\{f, d_1\}$
d_1	λ	λ	h_1	λ	(h_1, h_1)	$\{f, d_2\}$
d_2					(λ, λ)	$\{f, m\}$
m					(h_1, λ)	$\{d_1, d_2\}$
						$\{d_1, m\}$
						$\{d_2, m\}$

Table 2

Note that the couple and h_1 have responsive preferences In what follows, we argue that there is no stable matching for the preference profile given in Table 2.

Assume for contradiction, that there exists a stable matching μ for the given preference profile. We look at the following allocations of the couple in μ.

(i) Suppose $\mu(c) = (\lambda, h_1)$.
Since $d_i \bar{P}_{h_1} m$, $r_1(d_i) = h_1$ for $i \in \{1, 2\}$, and $\kappa_{h_1} = 2$, therefore $d_i \notin \mu(h_1)$ for some i. Thus, stability of μ implies that (h_1, d_i) blocks μ.

(ii) Suppose $\mu(c) = (\lambda, \lambda)$.
Since $\{f, m\} P_{h_1} \{d_1, d_2\}$ and $(h_1, h_1) P_c(\lambda, \lambda)$, $((h_1, h_1), c)$ blocks μ.

(iii) Suppose $\mu(c) = (h_1, h)$ for some $h \in \{h_1, \lambda\}$.
Since $r_1(P_f) = \lambda$, it follows by responsiveness that $((\lambda, h), c)$ blocks μ.

Since cases (i)-(iii) are exhaustive, it follows that there is no stable matching for the given preference profile in Table 2.

4 Conditions on couples' preferences for stability

In view of Example 1, we look for necessary and sufficient conditions on couples' preferences that guarantee the existence of a stable matching for every profile.

Let $P_C^0 = (\{P_c^0\}_{c \in C})$ be a given profile of preferences of the couples. Thus, for any $c = \{f, m\} \in C$, and a given couple preference P_c^0, P_f^0 and P_m^0 denote the individual preferences of f and m respectively. Since preferences of couples are responsive, a couple preference P_c^0 uniquely determines the individual preferences P_f^0 and P_m^0 of the members of the couple. In what follows, we present a condition on P_C^0 called extreme-altruism.

DEFINITION 8 *A profile of couple preferences P_C^0 is said to satisfy extreme-altruism if for all $c = \{f, m\} \in C$, all $h \in H$ and all $h', h'' \in H \cup \{\lambda\}$:*

(i) *$h' P_f^0 h$, $h' R_f^0 h''$, $h'' R_m^0 \lambda$ and $\kappa_h \le |D| - 2$ imply $(h', h'') P_c^0(h, h)$, and*

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(ii) $h'P_m^0 h$, $h'R_m^0 h''$, $h''R_f^0\lambda$ and $\kappa_h \leq |D| - 2$ imply $(h'',h')P_c^0(h,h)$.

For illustration of extreme-altruism, consider a couple $c = \{f,m\}$ and hospitals $h \in H$ and $h', h'' \in H \cup \{\lambda\}$. Suppose f strictly prefers h' to h and weakly prefers h' to h'', and m weakly prefers h'' to λ. Then extreme-altruism says that, if h does not have sufficiently large capacity, then c ranks (h',h'') above (h,h). Note here, that if h has a sufficiently large capacity, then we would not require any restriction on couples' preferences, simply because the hospital h will have enough vacancies to accommodate at least one member of the couple, thereby removing the possibility of the couple to block with the hospital.

Out of two allocations of a couple, one where both the members are allocated at the same hospital and the other where one member moves to a better hospital (according to his/her individual preference), the couple prefers the latter to the former. For instance, if the hospital h' gives a substantially better salary compared to h. Then the couple would rather prefer on member to be at h' than both the members of the couple to be at h.

Our next lemma shows that extreme-altruism and responsiveness together imply that if two hospitals are acceptable for both members of a couple, then the ranking of those two hospitals by each member of the couple is always the same. That is, if h_1 and h_2 are both acceptable by each member of a couple $\{f,m\}$, then f and m will have same ranking over h_1 and h_2.

LEMMA **6** *Let P_C^0 be a profile of couple preferences satisfying extreme-altruism. Then, for any $c = \{f,m\}$ and any $h_1, h_2 \in H$ such that $hP_x^0\lambda$ for all $x \in \{f,m\}$ and all $h \in \{h_1, h_2\}$, we have $h_1 P_f^0 h_2$ if and only if $h_1 P_m^0 h_2$.*

Proof: Let P_C^0 be a profile of couple preferences satisfying extreme-altruism. Consider a couple $c = \{f,m\}$ with preference P_c^0 and consider two hospitals $h_1, h_2 \in H$. Assume for contradiction $h_1 P_f^0 h_2 P_f^0 \lambda$ and $h_2 P_m^0 h_1 P_m^0 \lambda$. By responsiveness and Condition (i) of the definition of extreme-altruism, we have $(h_1,\lambda)P_c^0(h_2,h_2)P_c^0(\lambda,h_2)$. Again, responsiveness and Condition (ii)

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of extreme-altruism implies $(\lambda, h_2)P_c^0(h_1, h_1)P_c^0(h_1, \lambda)$. However, this is a contradiction. $\square$

Our next corollary follows directly from Lemma 6. It says the following. Suppose P_C^0 satisfies the extreme-altruism. Consider a couple. Suppose that every hospital is acceptable for each member of the couple. Then, the members of the couple have the same individual preference over H.

COROLLARY 1 *Let P_C^0 be a profile of couple preferences satisfying extreme-altruism. Let $c = \{f, m\} \in C$ be a couple such that $hP_f^0\lambda$ and $hP_m^0\lambda$ for all $h \in H$. Then $P_m^0 = P_f^0$.*

For a profile of preferences P_C^0 of the couples, an extension of P_C^0 is defined as a preference profile $P = (\{P_d\}_{d \in D}, \{P_c\}_{c \in C}, \{P_h\}_{h \in H})$ such that $P_c = P_c^0$ for all $c \in C$.

The following theorem says that extreme-altruism of P_C^0 is necessary and sufficient for the existence of a stable matching for *every* extension of P_C^0.

THEOREM 1   *(i) If P_C^0 satisfies extreme-altruism, then a stable matching exists for any extension of P_C^0.*

(ii) If P_C^0 does not satisfy extreme-altruism for all $h \in H$, then there exists an extension of P_C^0 with no stable matching.

Proof: [Part (i)] Consider a preference profile P that is an arbitrary extension of P_C^0 where P_C^0 satisfies extreme-altruism. We show that the DPDA where each member of each couple proposes according to his/her individual preference gives a stable matching for P.

Let μ be the outcome. Suppose μ is not stable at P. Since DPDA is individually rational, Lemma 1, Lemma 2 and Lemma 3 imply that μ is blocked by $((h, h), c)$ for some $h \in H$ and some $c = \{f, m\} \in C$. Let $\mu(f) = h_f$ and $\mu(m) = h_m$. Thus $(h, h)P_c^0(h_f, h_m)$.

Assume without loss of generality that $fP_h m$. By Lemma 4 and Lemma 5, we know that $h_f P_f^0 h$. Thus, by responsiveness, we must have $hP_m^0 h_m$.

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Suppose $\kappa_h > |D| - 2$. Then $\{f, m\} \notin \mu(h)$ implies $|\mu(h)| \le |D| - 2 < \kappa_h$. But since $h P_m^0 h_m$, m proposed to h at an earlier step of DPDA and got rejected even when h had a vacancy. Thus $m \notin D_h$ which implies that $((h, h), c)$ cannot block μ. Thus $\kappa_h \le |D| - 2$.

By the definition of DPDA, we have $h_f R_f^0 \lambda$ and $h_m R_m^0 \lambda$. Also, we know that x is weakly preferred to itself. This together with extreme-altruism implies $(h_f, \lambda) P_c^0(h, h) P_c^0(h_f, h_m)$.

This contradicts the fact that $h_m R_m^0 \lambda$. Thus $(h_f, h_m) R_c^0(h_f, \lambda) P_c^0(h, h)$ which contradicts that $((h, h), c)$ blocks μ. This completes the proof of part (i) of Theorem 1.

[Part (ii)] Suppose P_C^0 does not satisfy extreme-altruism. We show that there exists an extension of P_C^0 with no stable matching.

Since P_C^0 does not satisfy extreme-altruism, there is a couple $c = \{f, m\}$, a hospital h such that $\kappa_h \le |D| - 2$ and hospitals $h_1, h_2 \in H \cup \{\lambda\} \setminus \{h\}$ such that:

(i) either, $h_1 P_f^0 h$, $h_1 R_f^0 h_2$ and $h_2 R_m^0 \lambda$, but $(h, h) P_c^0(h_1, h_2)$,

(ii) or, $h_1 P_m^0 h$, $h_1 R_m^0 h_2$ and $h_2 R_f^0 \lambda$, but $(h, h) P_c^0(h_2, h_1)$.

Assume without loss of generality that (i) holds.

By responsiveness, $h_1 P_f^0 h$ and $(h, h) P_c^0(h_1, h_2)$, implies $h P_m^0 h_2$. Consider a preference profile P such that

(i) for all $h' \in H \setminus \{h, h_1, h_2\}$, either $|\{d : d P_h c \text{ and } r_1(P_d) = h\}| = \kappa_{h'}$ or $f, m \notin D_{h'}$,

(ii) there are doctors $d_1, d_2 \in D \setminus \{f, m\}$ such that $f P_h d_1 P_h d_2 P_h m$ and $\{f, m\} P_h \{d_1, d_2\}$,

(iii) $|\{d : d P_h f \text{ and } r_1(P_d) = h\}| = \kappa_h - 2$. This is possible since $\kappa_h \le |D| - 2$ implies $\kappa_h - 2 \le |D| - 4$,

(iv) either $f P_{h_1} d_1 P_{h_1} m$ and $|\{d : d P_{h_1} f \text{ and } r_1(P_d) = h_1\}| = \kappa_{h_1} - 1$, or $r_1(P_{h_1}) = f$, $r_2(P_{h_1}) = d_1$ and $m \notin D_{h_1}$,

(v) either $|\{d : dP_{h_2}c$ and $r_1(P_d) = h_2\}| = \kappa_{h_2} - 1$ or $f \notin D_{h_2}$,

(vi) $r_1(P_{d_1}) = h_1$ and $r_2(P_{d_1}) = r_1(P_{d_2}) = h$.

But it trivially follows from this preference profile that for a stable matching μ, $h' \notin \{\mu(f), \mu(m)\}$ for $h' \notin \{h, h_1, h_2\}$. Also, it is not possible that $\mu(c) = (h_i, h_i)$ for $i \in \{1, 2\}$. Thus, by our construction, a stable matching exists for this matching problem if and only if there is a stable matching for Example 1. However, since there does not exist a stable matching for Example 1, thus, we do not have a stable matching for P. This completes the proof of part (ii) of Theorem 1. ∎

From the above example, it is clear that even if all hospitals view all the doctors as acceptable, violation of extreme-altruism can lead to a preference profile with no stable matching, if we have enough doctors to fulfil the capacity constraints as given by points (i),(ii), (iv) and (v). Thus we get the following corollary.

COROLLARY 2 *Suppose $D_h = D$ for all $h \in H$. Moreover, $\sum_{h \in H} \kappa_h = |D|$, then:*

(i) *If P_C^0 satisfies extreme-altruism, then a stable matching exists for any extension of P_C^0.*

(ii) *If P_C^0 does not satisfy extreme-altruism for all $h \in H$, then there exists an extension of P_C^0 with no stable matching.*

5 Conditions on hospitals' preferences for stability

In Section 4, we have discussed a necessary and sufficient condition on couples' preferences that guarantees the existence of a stable matching for every collection of preferences of the hospitals. In this section, we look at the other

side of the problem, that is, we look for necessary and sufficient condition on hospitals' preferences so that a stable matching exists for every collection of preferences of the doctors (both individuals and couples).

Let $P_H^0 = (\{P_h^0\}_{h \in H})$ be a given profile of preferences of the hospitals. In what follows, we introduce the *aversion to couple diversity* property.

DEFINITION **9** *A profile of hospital preferences P_H^0 is said to have aversion to couple diversity if for all $h \in H$, all $c = \{f, m\}$ and all $d_1, d_2 \in D$ with $f, m \in D_h$ such that either (i) $fP_h^0 d_1 P_h^0 d_2 P_h^0 m$ and $|\{d : dP_h^0 m\}| > \kappa_h$, or (ii) $mP_h^0 d_1 P_h^0 d_2 P_h^0 f$ and $|\{d : dP_h^0 f\}| > \kappa_h$, we have $\{d_1, d_2\}P_h^0\{f, m\}$.*

Consider a hospital h with a preference P_h over acceptable and feasible sets of doctors. Take a couple $c = \{f, m\}$ such that both f and m are acceptable for h but at least one of them is not amongst the top-κ_h doctors according to the restriction of P_h over individual doctors. Suppose that there are two doctors d_1, d_2 who are ranked in-between f and m according to P_h. Then, aversion to couple diversity says that the set $\{d_1, d_2\}$ must be preferred to the couple c according to P_h. Note here that if both the members of the couple are in the top-κ_h doctors according to the restriction of P_h over individual doctors, then we do not need this condition as for a stable matching, the couple will always be a part of h.

So, in other words, whenever a hospital compares a couple and another pair of doctors over which responsiveness does not induce the comparison, the hospital prefers the couple only if at most one doctor from the other pair ranks in-between the members of the couple. Thus, a hospital has aversion to couple diversity if it does not like to employ a couple whose members have relatively more dissimilar ranking in its preference.

It is important to note here that the diversity aversion just applies to couples and not single doctors as two single doctors can not apply to a hospital together and block a matching. On the other hand, we can encounter a situation where a couple applies to a hospital such that a member of a couple is individually worse off but the couple is better off as a whole.

For a profile of preferences P_H^0 of the hospitals, an extension of P_H^0 is defined as a preference profile $P = (\{P_d\}_{d\in D}, \{P_c\}_{c\in C}, \{P_h\}_{h\in H})$ such that $P_h = P_h^0$ for all $h \in H$.

Our next theorem says that the aversion to couple diversity of P_H^0 is necessary and sufficient for the existence of a stable matching for every extension of P_H^0.

THEOREM **2**   *(i) If P_H^0 satisfies aversion to couple diversity property, then a stable matching exists at every extension of P_H^0.*

(ii) If P_H^0 does not satisfy aversion to couple diversity property, then there exists an extension of P_H^0 with no stable matching.

Proof: [Part (i)] Consider a preference profile P that is an arbitrary extension of P_H^0 where P_H^0 satisfies aversion to couple diversity. We show that the DPDA where each member of each couple proposes according to his/her individual preference gives a stable matching for P.

Let μ be the outcome. Suppose μ is not stable at P. Since DPDA is individually rational, Lemma 1, Lemma 2 and Lemma 3 imply that μ is blocked by $((h,h),c)$ for some $h \in H$ and some $c = \{f,m\} \in C$. Let $\mu(f) = h_f$ and $\mu(m) = h_m$. Thus $(h,h)P_c^0(h_f, h_m)$.

Assume without loss of generality that fP_hm. If $m \notin D_h$, then $((h,h),c)$ cannot block μ as it violates individual rationality. Thus $mP_h^0\lambda$. By Lemma 4 and Lemma 5, we know that $h_fP_f^0h$. Thus, responsiveness implies hP_mh_m.

It follows that, before applying to h_m, m applied to h and got rejected. Therefore, $|\{d : dP_h^0m \text{ and } d \in \mu(h)\}| = \kappa_h$. Since fP_h^0m and $f \notin \mu(h)$, we have $|\{d : dP_h^0f\}| > \kappa_h$. This, together with aversion to couple diversity, implies that $\{d,d'\}P_h^0\{f,m\}$ for all $d, d' \in \mu(h)$, which is a contradiction to the fact that $((h,h),c)$ blocks μ. This completes the proof of part (i) of Theorem 2.

[Part (ii)] Suppose P_H^0 does not satisfy the aversion to couple diversity property. We show that there is an extension of P_H^0 with no stable matching.

Since P_H^0 does not satisfy the aversion to couple diversity property, we have $h \in H$, $c = \{f, m\} \in C$ and $d_1, d_2 \in D$ with $f, m \in D_h$ such that $\{f, m\} P_h^0 \{d_1, d_2\}$ and either

(i) $f P_h^0 d_1 P_h^0 d_2 P_h^0 m$ and $|\{d : d P_h^0 m\}| > \kappa_h$, or

(ii) $m P_h^0 d_1 P_h^0 d_2 P_h^0 f$ and $|\{d : d P_h^0 f\}| > \kappa_h$.

Assume without loss of generality that (i) holds. Consider a preference profile P such that

(i) $\lambda P_f h$,

(ii) $h P_m \lambda$,

(iii) $(h, h) P_c (\lambda, \lambda)$,

(iv) for all $h' \in H \setminus \{h\}$ either $h P_f h'$, or $|\{d : d P_{h'}^0 f \text{ and } r_1(P_d) = h'\}| = \kappa_{h'}$,

(v) for all $h' \in H \setminus \{h\}$ either $\lambda P_m h'$, or $|\{d : d P_{h'}^0 m \text{ and } r_1(P_d) = h'\}| = \kappa_{h'}$,

(vi) $r_1(P_{d_1}) = r_1(P_{d_2}) = h$, and

(vii) $|\{d : d P_h^0 m \text{ and } r_1(P_d) = h\}| = \kappa_h$. Note, that this also includes d_1 and d_2.

But it trivially follows from the given preference profile that for a stable matching μ, for all $h' \in H \setminus \{h\}$, $h' \notin \{\mu(f), \mu(m)\}$. Thus, by our construction, a stable matching exists for this matching problem if and only if there is a stable matching for Example 2. However, since there does not exist a stable matching for Example 2, thus we do not have a stable matching for P. This completes the proof of part (ii) of Theorem 2. ∎

From the example above, it is not clear if non-aversion to couple diversity can always lead to a preference profile with no stable matching when doctors prefer to be matched any hospital than being unemployed. We show by the means of an example that if all the doctors are averse to unemployment, then

we can not always obtain an extension of P_H^0 with no stable matching, when P_H^0 does not satisfy aversion to couple diversity.

EXAMPLE **3** Consider a matching problem with $H = \{h_1, h_2\}$ and $D = \{f, m, s_1, s_2\}$ such that $c = \{f, m\}$ is the only couple. Let $\kappa_{h_1} = \kappa_{h_2} = 2$. Thus, $\sum_{h \in H} \kappa_h = |D|$. The preferences of hospitals on individual doctors is given in the table below. The doctors prefer to be matched to any hospital than being unemployed.

P_{h_1}	P_{h_2}
f	s_2
s_1	m
s_2	f
m	s_1

Let $\{f, m\} P_{h_1} \{s_1, s_2\}$. Thus the preference h_1 does not follow aversion to couple diversity.

We show that there exists a stable matching for these preferences of hospitals for any preferences of the doctors and the couple.

Let μ be a matching for the given preferences of the hospitals such that $\mu(h_1) = \{f, s_1\}$ and $\mu(h_2) = \{s_2, m\}$. Clearly, h_1 and h_2 have their top ranked doctors. Thus, neither h_1 nor h_2 would like to block μ with any other doctor. Also, no doctor would block μ with λ as all doctors prefer being matched to any hospital than be unemployed. Thus μ is stable for any preferences of doctors even when there is no aversion for diverse couples.

The above example leads to the following corollary.

COROLLARY **3** *Suppose $hP_d\lambda$ for all $h \in H$ and all $d \in D$, then a stable matching always exists for any extension of P_H^0 when P_H^0 satisfies aversion to diverse couples.*

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6 Concluding remarks

As we have discussed earlier, existence of a stable matching is guaranteed at a preference profile if it satisfies responsiveness as defined in Klaus and Klijn[9]. The different result in our paper stems from the fact that we allow for a setwise blocking notion, which allows for the existence of more blocking coalitions. Here, a hospital is allowed to replace two doctors by a couple, whilst Klaus and Klijn consider pairwise blocking. Thus, if a couple wishes to block with a hospital, both the members of the couple will be considered separately by the hospital instead of considering the couple as a whole. We explain this in detail in the following paragraph.

Consider Example 1 and Example 2. According to the model in Klaus and Klijn[9], given the preference $\bar{P}_{h_1}$, the pair $\{f, m\}$ can not block with h_1 to remove the pair of doctors $\{d_1, d_2\}$. By their blocking notion, each member of the couple can only replace a doctor who is ranked lower to that member of the couple. The fact that d_1 and d_2 are ranked in-between f and m, prevent the couple to block with h_1.

Now, consider the matching μ_1 for Example 1 and μ_2 for Example 2 such that $\mu_1(c) = (h_2, h_3)$, $\mu_1(d_1) = \mu_1(d_2) = h_1$ and $\mu_2(c) = (\lambda, \lambda)$, $\mu_2(d_1) = \mu_2(d_2) = h_1$. Note that, in our model, both these matchings are blocked by $((h_1, h_1), c)$. However, this block is *not* possible according to the model in Klaus and Klijn[9]. It can be verified that μ_1 and μ_2 are indeed stable according to their model.

In this paper, we have shown that the existence of a stable matching is not guaranteed when couples and/or hospitals have complete and responsive preferences. We have provided (a) necessary and sufficient conditions on couples' preferences so that a stable matching exists at every extension of those preferences, and (b) necessary and sufficient conditions on hospitals' preferences so that a stable matching exists at every extension of those preferences. Additionally, we have provided algorithms that produce a stable matching whenever that exists in this framework.

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