Local Global Equivalence for Unanimous Social Choice Functions*

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Abstract

We identify a condition on preference domains that ensures that every locally strategy-proof and unanimous random social choice function is also strategy-proof. Furthermore every unanimous, locally strategy-proof deterministic social choice function is also group strategy-proof. The condition identified is significantly weaker than the characterization condition for local-global equivalence without unanimity in Kumar et al. (2020). The condition is not necessary for equivalence with unanimous random/deterministic social choice functions. However, we show the weaker condition of connectedness remains necessary.

Keywords: Local strategy-proofness; strategy-proofness; unanimity

JEL Classification: D71.

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1 Introduction

The theory of mechanism design investigates the objectives that can be achieved by a group of agents (or a planner) when these objectives depend on information held privately by the agents. Agents must be induced to reveal their private information truthfully: in more formal terms, the Random/Deterministic Social Choice Function (RSCF/DSCF) representing the objectives of the planner must be *incentive compatible* or *strategy-proof.* A RSCF/DSCF is strategy-proof if no agent can gain by misrepresenting her preferences irrespective of the preference announcements of the other agents. In particular, in the random setting, we use the stochastic dominance notion for strategy-proofness. In many contexts, it is plausible to assume that an agent can only misrepresent to a "local" preference. The class of locally strategy-proof RSCFs should, in principle be larger than the class of strategy-proof RSCFs. However, Carroll (2012) and Sato (2013) demonstrate that for many important preference domains and a natural notion of localness (adjacency), the classes of locally strategy-proof and strategy-proof RSCFs/DSCFs coincide. We shall refer to this property as local-global equivalence. This property has important theoretical and practical implications which are discussed in both papers.

Kumar et al. (2020) formulate the local-global equivalence problem more generally, in the context of an "environment". An environment is a graph where the nodes represent admissible preferences and the edges, the notion of localness. They characterize environments that satisfy local-global equivalence. The necessary and sufficient condition for local-global equivalence requires the existence of certain kinds of paths in the graph. An important aspect of the paper is that it considers a single-agent model. Our goal in this paper is to show that in a multi-agent problem, a much weaker condition is sufficient, when the set of RSCFs under consideration satisfy the familiar and mild efficiency property of *unanimity.* We note that imposing unanimity in a single-agent model renders it trivial — it is an interesting requirement only in a multi-agent problem.[1]

We consider a model with a finite number of alternatives. A preference *domain* is a collection of strict orderings of the alternatives. A pair of preferences is local if there is a single pair

[1]Formally, the models in Kumar et al. (2020), Carroll (2012) and Sato (2013) are also multi-agent models. Since they do not impose unanimity, the multi-agent model is indistinguishable from its single-agent counterpart. For this reason, we choose to refer to the models in these papers as single-agent models.

of alternatives whose ranking is reversed between the two preferences.[2] We consider RSCFs that satisfy unanimity, i.e. those that respect consensus amongst agents. A domain satisfies equivalence if every unanimous locally strategy-proof RSCF is also strategy-proof.

In this setting, we show that a condition first identified in Sato (2013) (which we refer to as Property P) has very important implications. This condition was shown to be necessary (but not sufficient) in the single-agent problem by Sato (2013). Property P is a weak condition, which specifies for every pair of alternatives, the existence of a path where preferences over this pair are not reversed more than once.[3] In contrast, the necessary and sufficient condition in Kumar et al. (2020) (which they call Property L) requires the existence of a path that satisfies no-restoration with respect to *all* alternatives in an appropriate lower contour set.[4]

We prove two main results using Property P. We show that it is sufficient for equivalence. In contrast, Kumar et al. (2020) show that the stronger Property L is not sufficient for local-global equivalence for RSCFs in the single-agent model.[5] Furthermore, a stronger result in the deterministic setting is true: every unanimous, locally strategy-proof DSCF on a domain satisfying Property P is also group strategy-proof. Our result is independent of the results in the existing literature on domains where strategy-proofness and group strategy-proofness are equivalent (see Section 4.1). Our overall conclusion is that imposing the requirement of unanimity leads to a considerable weakening of the conditions required for equivalence in both random and deterministic settings.

As mentioned earlier, Property P is a weak condition. It is satisfied by several familiar domains such as the universal domain and the single-peaked domain. However it is not a necessary condition for a domain to satisfy equivalence of local strategy-proofness and strategy-proofness for unanimous RSCFs/DSCFs. In Section 4.1, we construct an example demonstrating this fact. We also show that the weaker condition of connectedness remains necessary for equivalence.

In recent work, Hong and Kim (2020) independently derive a condition slightly weaker than our Property P and show that it is sufficient for equivalence.[6] They focus on ordinal Bayesian

[2] This is the "adjacency" notion of localness used in Carroll (2012) and Sato (2013).

[3] Further discussion of domains satisfying Property P can be found in Section 3.

[4] Sato (2013) and Carroll (2012) also provide stronger sufficient conditions for equivalence in the single-agent model.

[5] Cho (2016) also considers the local-global equivalence issue for RSCFs in the single-agent model. The paper provides sufficient conditions for a variety of lottery comparisons.

[6] The two conditions are equivalent if the domain satisfies the following richness property: for every alter-

incentive compatible DSCFs and dictatorial domains. In contrast, we study RSCFs and extend our result for DSCFs to cover group strategy-proofness. We discuss their condition further in Section 4.1 where we also show that it, like Property P, is not necessary for equivalence.

The paper is organized as follows. Section 2 describes the model. Section 3 introduces and discusses Property P, which is the key condition for our results. The main results are in Section 4 while Section 5 discusses issues regarding necessity.

2 The Model

Let $A = \{a, b, \dots\}$ denote a finite set of alternatives with $|A| \geq 3$. Let $N = \{1, 2, \dots, n\}$ denote a finite set of voters with $n \geq 2$. A preference P_i of voter i is an antisymmetric, complete and transitive binary relation over A, i.e. a *linear order*. Given $a, b \in A$, aP_ib is interpreted as "a is strictly preferred to b" according to P_i. Let $r_k(P_i)$, $k = 1, \dots, |A|$ denote the k^{th} ranked alternative in preference P_i, i.e. $[r_k(P_i) = a] \Leftrightarrow \big[\big|\{x \in A : xP_ia\}\big| = k - 1\big]$. Let $\mathcal{P}$ denote the set of all preferences - the set $\mathcal{P}$ will be referred to as the *universal domain*. We shall refer to an arbitrary set $\mathcal{D} \subseteq \mathcal{P}$ as a *domain*.[7] A preference profile is an n-tuple $P = (P_1, P_2, \dots, P_n)$.

Fix a pair of preferences $P_i, P'_i \in \mathcal{D}$. Two alternatives a and b are *reversed* between P_i and P'_i if aP_ib and bP'_ia, or bP_ia and aP'_ib hold. Accordingly, two preferences P_i and P'_i are *adjacent/local*, denoted by $P_i \sim P'_i$, if there exists exactly one pair of alternatives that are reversed between P_i and P'_i; formally, there exists $1 \leq k < |A|$ such that $r_k(P_i) = r_{k+1}(P'_i)$, $r_k(P'_i) = r_{k+1}(P_i)$ and $r_l(P_i) = r_l(P'_i)$ for all $l \notin \{k, k+1\}$. A *path* $\pi \equiv (P_i^1, \dots, P_i^t)$ is a sequence of non-repeated preferences in $\mathcal{D}$ satisfying the property that consecutive preferences are adjacent, i.e. $P_i^k \sim P_i^{k+1}$ for all $k = 1, \dots t-1$. The set of all paths from P_i to P'_i where $P_i, P'_i \in \mathcal{D}$ is denoted by $\Pi(P_i, P'_i)$. The domain $\mathcal{D}$ is *connected* if there exists a path between every pair $P_i, P'_i \in \mathcal{D}$.

Our model is identical to the models in Sato (2013) and Carroll (2012). It is a special case of the model in Kumar et al. (2020) where the notion of localness is completely general. On the other hand, we consider a many-agent setting while Kumar et al. (2020) only consider the single-agent problem.

Let $\Delta(A)$ denote the set of probability distributions over A. An element $\lambda \in \Delta(A)$ will

native a, there exists a preference in the domain whose first-ranked alternative is a.

[7] We assume that all voters have the same preference domain $\mathcal{D}$.

be referred to as a *lottery*. We let λ_a denote the probability with which $a \in A$ is selected by λ. Thus $0 \leq \lambda_a \leq 1$ and $\sum_{a \in A} \lambda_a = 1$. Given a preference P_i, the lottery λ *stochastically dominates* lottery λ' according to P_i (denoted by $\lambda P_i^{sd} \lambda'$) if $\sum_{k=1}^{t} \lambda_{r_k(P_i)} \geq \sum_{k=1}^{t} \lambda'_{r_k(P_i)}$ for all $1 \leq t \leq |A|$.

OBSERVATION **1** Fix P_i and $\lambda, \lambda' \in \Delta(A)$ such that $\lambda P_i^{sd} \lambda'$. Pick $a, b \in A$ such that aP_ib. Let $\hat{\lambda} \in \Delta(A)$ be such that (i) $\hat{\lambda}_b > \lambda'_b$, (ii) $\hat{\lambda}_a < \lambda'_a$ and (iii) $\hat{\lambda}_c = \lambda'_c$ for all $c \notin \{a, b\}$. Then $\lambda P_i^{sd} \hat{\lambda}$. The lottery $\hat{\lambda}$ is obtained by transferring probability weight from an alternative a to a less preferred one b, in λ' while keeping all other probabilities unchanged. It is easy to verify that $\lambda' P_i^{sd} \hat{\lambda}$ from which $\lambda P_i^{sd} \hat{\lambda}$ follows immediately.

DEFINITION **1** *A Random Social Choice Function (RSCF) is a map* $\varphi : \mathcal{D}^n \to \Delta(A)$.

Given $a \in A$, let $\varphi_a(P)$ denote the probability with which a is selected at the profile P. A Deterministic Social Choice Function (DSCF) $f : \mathcal{D}^n \to \Delta(A)$ is a particular RSCF such that for each $P \in \mathbb{D}^n$, $f_a(P) = 1$ for some $a \in A$. Henceforth, for ease of presentation, we write a DSCF as $f : \mathbb{D}^n \to A$, where an alternative is selected at each preference profile.

We require all RSCFs under consideration to satisfy the property of *unanimity*. This is a weak form of efficiency where the RSCF selects a commonly first-ranked alternative with probability 1 whenever it exists.

DEFINITION **2** *A RSCF* $\varphi : \mathcal{D}^n \to \Delta(A)$ *is unanimous if for all* $P \in \mathcal{D}^n$,

$$[r_1(P_i) = a \text{ for all } i \in N] \Rightarrow [\varphi_a(P) = 1].$$

Correspondingly, a DSCF $f : \mathcal{D}^n \to A$ is *unanimous* if for all $P \in \mathcal{D}^n$, we have $[r_1(P_i) = a$ for all $i \in N] \Rightarrow [f(P) = a]$. In order to avoid trivial considerations, we assume throughout that $\mathcal{D}$ contains at least two preferences with distinct peaks.

A RSCF is locally strategy-proof if a voter cannot gain by a misrepresentation to an adjacent preference (in other words, according to the sincere preference, the social lottery induced by any misrepresentation to an adjacent preference is always stochastically dominated by the lottery delivered by truthtelling). On the other hand, a RSCF is strategy-proof if a voter cannot gain by an arbitrary misrepresentation.

DEFINITION **3** *A RSCF* $\varphi : \mathcal{D}^n \to A$ *is locally manipulable by an agent* $i \in N$ *at profile* $P = (P_i, P_{-i})$ *if there exists* $P'_i \in \mathcal{D}$ *with* $P_i \sim P'_i$ *such that* $\varphi(P_i, P_{-i}) P_i^{sd} \varphi(P'_i, P_{-i})$ *does not* *hold, i.e.* $\sum_{k=1}^{t} \varphi_{r_k(P_i)}(P_i, P_{-i}) < \sum_{k=1}^{t} \varphi_{r_k(P_i)}(P'_i, P_{-i})$ *for some* $1 \leq t < |A|$. *The RSCF* φ *is locally strategy-proof if it is not locally manipulable by any agent at any profile.*

DEFINITION **4** *A RSCF* $\varphi : \mathcal{D}^n \to A$ *is manipulable by an agent* $i \in N$ *at profile* $P = (P_i, P_{-i})$ *if there exists* $P'_i \in \mathcal{D}$ *such that* $\varphi(P_i, P_{-i}) P_i^{sd} \varphi(P'_i, P_{-i})$ *does not hold, i.e.* $\sum_{k=1}^{t} \varphi_{r_k(P_i)}(P_i, P_{-i}) < \sum_{k=1}^{t} \varphi_{r_k(P_i)}(P'_i, P_{-i})$ *for some* $1 \leq t < |A|$. *The RSCF* φ *is strategy-proof if it is not manipulable by any agent at any profile.*

A strategy-proof RSCF is clearly locally strategy-proof. We investigate the structure of domains where the converse is true for *all* unanimous RSCFs.

DEFINITION **5** *The domain* $\mathcal{D}$ *satisfies local-global equivalence for unanimous RSCFs (uLGE) if every unanimous and locally strategy-proof RSCF* $\varphi : \mathcal{D}^n \to \Delta(A)$, $n \geq 2$, *is strategy-proof.*

We can correspondingly define local-global equivalence for DSCFs. A DSCF $f : \mathcal{D}^n \to A$ is *locally strategy-proof* (respectively, *strategy-proof*) if for all $i \in N$, $P_i, P'_i \in \mathcal{D}$ with $P_i \sim P'_i$ (respectively, $P_i, P'_i \in \mathcal{D}$) and $P_{-i} \in \mathcal{D}^{n-1}$, either $f(P_i, P_{-i}) = f(P'_i, P_{-i})$ or $f(P_i, P_{-i}) P_i f(P'_i, P_{-i})$ holds. The domain $\mathcal{D}$ satisfies local-global equivalence for unanimous DSCFs if every unanimous and locally strategy-proof DSCF $f : \mathcal{D}^n \to A$, $n \geq 2$, is strategy-proof.

In the next section, we provide a sufficient condition for uLGE.

3 A Sufficient Condition

In this section, we introduce Property P that is central to our results. Let $\mathcal{D}$ be a domain and $a, b \in A$ be a pair of alternatives. A path $\pi = (P_i^1, \ldots, P_i^t)$ satisfies *no* $\{a, b\}$*-restoration* if the relative ranking of a and b is reversed *at most* once along π, i.e. there does not exist integers q, r and s with $1 \leq q < r < s \leq t$ such that either (i) $a P_i^q b, b P_i^r a$ and $a P_i^s b$, or (ii) $b P_i^q a, a P_i^r b$ and $b P_i^s a$.[8]

Sato (2013) introduces the *pairwise no-restoration property.* This property requires that for every pair of distinct preferences and a pair of alternatives, there exists a path between the preferences that satisfies no-restoration with respect to the pair of alternatives.

[8]It is worth emphasizing that in our definition of "$\{a, b\}$-restoration", we are *not* referring to an ordered pair $\{a, b\}$. Thus $\{a, b\}$-restoration and $\{b, a\}$-restoration are the same in our definition.

Definition 6 *The domain $\mathcal{D}$ satisfies the pairwise no-restoration property (Property P) if for all distinct $P_i, P'_i \in \mathcal{D}$ and distinct $a, b \in A$, there exists a path $\pi = (P_i^1, \ldots, P_i^t) \in \Pi(P_i, P'_i)$ with no $\{a, b\}$-restoration.*

Property P is satisfied by the universal domain and the domain of single-peaked preferences. Conversely, Chatterji et al. (2021) show that any domain satisfying Property P and some additional regularity conditions must either be a sub-domain of the domain of single-peaked preferences or a *hybrid* domain which is a "perturbation" of the single-peaked domain. Alternatives are again ordered as in the single-peaked domain. Alternatives are partitioned into three segments, left, middle and right. A hybrid domain consists of all preferences satisfying the following property: preferences in the left and right segments are single-peaked while being unrestricted in the middle segment. Hybrid domains cover the universal domain and the single-peaked domain as special cases, the former in the case where the middle segment is the entire set of alternatives and the latter where the middle segment is the null set.

Sato (2013) shows that Property P is necessary but not sufficient for the equivalence of local strategy-proofness and strategy-proofness for DSCFs (henceforth called LGE) in a single-agent model (or equivalently without imposing unanimity). Kumar et al. (2020) formulate the *lower contour set no-restoration property* (Property L) that is necessary and sufficient condition for LGE in a more general model. Property L is satisfied if for all $P_i, P'_i \in \mathcal{D}$ and $a \in A$, there exists a path $\pi = (P_i^1, \ldots, P_i^t) \in \Pi(P_i, P'_i)$ with no $\{a, b\}$-restoration for all $b \in L(a, P_i) = \{z \in A : aP_iz\}$.

Property P is a weaker than Property L. This is illustrated in the example below which is adapted from Example 3.2 in Sato (2013).

Example 1 Let $A = \{x, y, z, u, v, w\}$. The domain $\mathcal{D}$ is specified in Table 1. Figure 1 shows all paths induced by the adjacent preferences in $\mathcal{D}$.

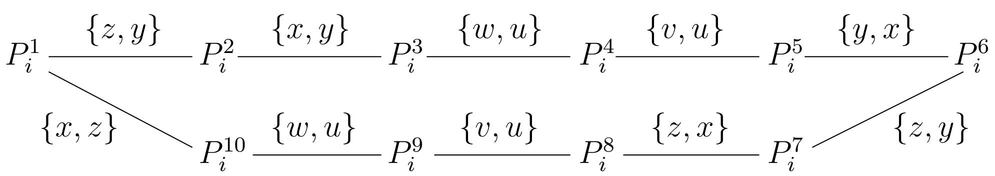

Figure 1: Paths induced by the adjacent preferences in $\mathcal{D}$

Figure 1 highlights an important property of $\mathcal{D}$ — there are exactly two paths between any pair of preferences. For example, between P_i^1 and P_i^7, there is a path $(P_i^1, P_i^2, P_i^3, P_i^4, P_i^5, P_i^6, P_i^7)$

P_i^1	P_i^2	P_i^3	P_i^4	P_i^5	P_i^6	P_i^7	P_i^8	P_i^9	P_i^{10}
x	x	y	y	y	x	x	z	z	z
z	y	x	x	x	y	z	x	x	x
y	z	z	z	z	z	y	y	y	y
v	v	v	v	u	u	u	u	v	v
w	w	w	u	v	v	v	v	u	w
u	u	u	w	w	w	w	w	w	u

Table 1: The Domain $\mathcal{D}$

and another path $(P_i^1, P_i^{10}, P_i^9, P_i^8, P_i^7)$. We shall refer to the former as the "clockwise" path and the latter as the "counter clockwise" path between P_i^1 and P_i^7. We shall in fact, refer to the clockwise and counter clockwise paths between any pair of preferences in $\mathcal{D}$. It can be verified that for any pair of distinct preferences and alternatives, either the clockwise path or the counter clockwise path is a path without restoration for the alternatives. Therefore, $\mathcal{D}$ satisfies Property P. However, it fails Property L, e.g. $z, y \in L(x, P_i^1)$, and the clockwise path from P_i^1 to P_i^7 has an $\{x, y\}$-restoration while the counter clockwise path from P_i^1 to P_i^7 has an $\{x, z\}$-restoration. We know there that LGE fails for $\mathcal{D}$. For instance, let $N = \{1, 2\}$ and consider the following DSCF:

$$f(P_1, P_2) = \begin{cases} z & P_1 = P_i^1, \\ y & P_1 = P_i^2, \text{ and} \\ r_1(P_1) & \text{otherwise.} \end{cases}$$

It is easy to verify that f is locally strategy-proof but fails strategy-proofness, e.g. $f(P_1^6, P_2) = x$, $f(P_1^1, P_2) = z$ and xP_1^1z.[9] It also violates unanimity, e.g. $f(P_1^1, P_2^2) = z \neq x$. Our result implies that *every* locally strategy-proof RSCF that fails to be strategy-proof on this domain must violate unanimity. Furthermore, every DSCF satisfying unanimity and local strategy-proofness is group strategy-proof. □

4 Main Results

Kumar et al. (2020) show that Property L does not guarantee that locally strategy-proof RSCFs are also strategy-proof. In this section, we show that this equivalence holds for unan-

[9]Here, (P_1^6, P_2) is a preference profile where agent 1's preference is P_i^6 and agent 2's preference is P_2 which is arbitrary. Similarly, (P_1^1, P_2) is a profile where agent 1's preference is P_i^1 and agent 2's preference is P_2.

imous RSCFs defined over domains satisfying the weaker Property P.

THEOREM 1 *If a domain satisfies Property P, it satisfies uLGE.*

Proof: Pick a domain $\mathcal{D}$ that satisfies Property P. Consider an arbitrary locally strategy-proof RSCF $\varphi : \mathcal{D}^n \to \Delta(A)$ that satisfies unanimity. We will show that φ is strategy-proof. We begin with an observation.

OBSERVATION 2 Consider $P_i, \bar{P}_i \in \mathcal{D}$ such that $P_i \sim \bar{P}_i$; in particular xP_iy and $y\bar{P}_ix$. If $\varphi(P_i, P_{-i}) \neq \varphi(\bar{P}_i, P_{-i})$ for some $P_{-i} \in \mathcal{D}^{n-1}$, then it must be the case that (i) $\varphi_y(\bar{P}_i, P_{-i}) > \varphi_y(P_i, P_{-i})$, (ii) $\varphi_x(\bar{P}_i, P_{-i}) < \varphi_x(P_i, P_{-i})$ and (iii) $\varphi_z(\bar{P}_i, P_{-i}) = \varphi_z(P_i, P_{-i})$ for all $z \notin \{x, y\}$. These properties are well-known in the literature. Gibbard (1977) refers to Parts (i) and (ii) as the property of being *non-perverse* and Part (iii) as the property of being *localized*.

LEMMA 1 *Let $P_i, \bar{P}_i \in \mathcal{D}$ be such that $P_i \sim \bar{P}_i$ and $r_1(P_i) = r_1(\bar{P}_i)$. Then $\varphi(P_i, P_{-i}) = \varphi(\bar{P}_i, P_{-i})$ for all $P_{-i} \in \mathcal{D}^{n-1}$.*

Proof: Assume w.l.o.g. that i is agent 1. Let $P_1, \bar{P}_1 \in \mathcal{D}$ be such that $P_1 \sim \bar{P}_1$ and $r_1(P_1) = r_1(\bar{P}_1) = a$. Let x, y be the alternatives that are reversed between P_1 and $\bar{P}_1$ with xP_1y and $y\bar{P}_1x$.

Let $P^k \equiv (P_1, P_2, \dots, P_k, P_1, \dots, P_1)$, i.e. P^k is the profile where agents 1 and $k+1, \dots, n$ have the preference P_1 while agents $2, \dots, k$ have preferences specified in the profile P_{-1}. Here $k \in \{1, \dots, n\}$ where $P^1 = (P_1, P_1, \dots, P_1)$ and $P^n = (P_1, P_2, \dots, P_n)$.

Let $\bar{P}^k \equiv (\bar{P}_1, P_2, \dots, P_k, P_1, \dots, P_1)$, i.e. $\bar{P}^k$ is the profile where agent 1 has the preference $\bar{P}_1$, agents $k+1, \dots, n$ have the preference P_1 and agents $2, \dots, k$ have preferences specified in the profile P_{-1}. Again $k \in \{1, \dots, n\}$ where $\bar{P}^1 = (\bar{P}_1, P_1, \dots, P_1)$ and $\bar{P}^n = (\bar{P}_1, P_2, \dots, P_n)$.

We will prove $\varphi(P^n) = \varphi(\bar{P}^n)$ by induction on k. Observe that $\varphi_a(P^1) = \varphi_a(\bar{P}^1) = 1$ since φ satisfies unanimity. Assume that $\varphi(P^{k-1}) = \varphi(\bar{P}^{k-1})$ for $k-1 < n$. We will show that $\varphi(P^k) = \varphi(\bar{P}^k)$.

We assume w.l.o.g. xP_ky. Since $\mathcal{D}$ satisfies Property P, there exists a path $(P_k^1, \dots, P_k^T) \in \Pi(P_1, P_k)$ such that xP_k^ry for all $r \in \{1, \dots, T\}$.

Let $P^{k,r} \equiv (P_1, P_2, \dots, P_{k-1}, P_k^r, P_1, \dots, P_1)$ and $\bar{P}^{k,r} \equiv (\bar{P}_1, P_2, \dots, P_{k-1}, P_k^r, P_1, \dots, P_1)$. The induction hypothesis implies that $\varphi(P^{k,1}) = \varphi(\bar{P}^{k,1})$. Suppose $\varphi(P^{k,T}) \neq \varphi(\bar{P}^{k,T})$. Let t be the smallest integer in the set $\{1, \dots, T\}$ such that $\varphi(P^{k,t}) \equiv \lambda \neq \bar{\lambda} \equiv \varphi(\bar{P}^{k,t})$. Clearly, $t > 1$. Observe that the profiles $P^{k,t}$ and $\bar{P}^{k,t}$ differ only in the preferences of agent 1 with

P_1 in the former profile and $\bar{P}_1$ in the latter. Thus, local strategy-proofness implies $\lambda P_1^{sd} \lambda'$ and then Observation 2 implies $\bar{\lambda}_y - \lambda_y > 0$, $\bar{\lambda}_x - \lambda_x < 0$ and $\bar{\lambda}_z = \lambda_z$ for all $z \notin \{x, y\}$. By the induction hypothesis, let $\varphi(P^{k,t-1}) = \varphi(\bar{P}^{k,t-1}) \equiv \lambda'$. Observe that the profiles $P^{k,t-1}$ and $P^{k,t}$ (respectively, profiles $\bar{P}^{k,t-1}$ and $\bar{P}^{k,t}$) differ only in the preferences of agent k being P_k^{t-1} in the former profile and P_k^t in the latter. Since $P_k^{t-1} \sim P_k^t$, local strategy-proofness implies that both λ and $\bar{\lambda}$ stochastically dominate λ' according to P_k^t, and λ' stochastically dominates both λ and $\bar{\lambda}$ according to P_k^{t-1}, and moreover there must be exactly one pair of alternatives which are reversed between P_k^{t-1} and P_k^t. This pair cannot be $\{x, y\}$ because $x P_k^r y$ for all P_k^r belonging to the path π. Suppose this pair is $\{a, x\}$ with $a \neq y$: in this case, by Part (iii) of Observation 2, $\lambda'_y = \lambda_y$ and $\lambda'_y = \bar{\lambda}_y$ contradicting our hypothesis that $\bar{\lambda}_y - \lambda_y > 0$. If the pair is $\{a, y\}$ with $a \neq x$, we contradict our assumption $\bar{\lambda}_x - \lambda_x < 0$. Similarly if the pair is $\{a, b\}$ with $a \neq x$ and $y \neq b$, we contradict both $\bar{\lambda}_y - \lambda_y > 0$ and $\bar{\lambda}_x - \lambda_x < 0$. This completes the proof. ■

Lemma **2** *Let $P \equiv (P_i, P_{-i}) \in \mathcal{D}^n$ be a profile and $a \in A$ be an alternative. Let $\bar{P}_i \in \mathcal{D}$ and suppose there exists a path $\pi = (P_i^1, \ldots, P_i^T) \in \Pi(P_i, \bar{P}_i)$ such that $a \neq r_1(P_i^k)$ for all $k \in \{1, \ldots, T\}$. Then $\varphi_a(P_i, P_{-i}) = \varphi_a(\bar{P}_i, P_{-i})$.*

Proof: Suppose the Lemma is false. Let $t \geq 2$ be the smallest integer in the set $\{1, \ldots, T\}$ such that $\varphi_a(P_i^{t-1}, P_{-i}) \neq \varphi_a(P_i^t, P_{-i})$. Consider the preferences P_i^{t-1} and P_i^t. If $r_1(P_i^{t-1}) = r_1(P_i^t)$, we have an immediate contradiction to Lemma 1. The remaining possibility is $r_1(P_i^{t-1}) \neq r_1(P_i^t)$. Here, there must be a reversal of the first and second ranked alternatives in P_i^{t-1} to obtain P_i^t. By assumption, a cannot be first or second ranked in either P_i^{t-1} or P_i^t; otherwise a would be ranked first in either P_i^{t-1} or P_i^t. Then, Part (iii) of Observation 2 implies $\varphi_a(P_i^{t-1}, P_{-i}) = \varphi_a(P_i^t, P_{-i})$ contradicting our initial assumption. ■

We can now complete the proof of the result. Let $P = (P_i, P_{-i})$ be a profile and $\bar{P}_i \in \mathcal{D}$. We will show $\varphi(P_i, P_{-i}) P_i^{sd} \varphi(\bar{P}_i, P_{-i})$. Pick an arbitrary path $\pi = (P_i^1, \ldots, P_i^t, \ldots, P_i^T) \in \Pi(P_i, \bar{P}_i)$. We will prove the result by induction on t.

The conclusion for the initial step ($t = 2$) follows from local strategy-proofness. Assume that $\varphi(P_i, P_{-i}) P_i^{sd} \varphi(P_i^{t-1}, P_{-i})$ for some $t > 2$. We will show $\varphi(P_i, P_{-i}) P_i^{sd} \varphi(P_i^t, P_{-i})$. If $\varphi(P_i^{t-1}, P_{-i}) = \varphi(P_i^t, P_{-i})$, then the result follows immediately. Assume therefore $\varphi(P_i^{t-1}, P_{-i}) \neq \varphi(P_i^t, P_{-i})$. Immediately, since $P_i^{t-1} \sim P_i^t$, by Lemma 1, it must be the case that $r_1(P_i^{t-1}) \equiv$

$a \neq b \equiv r_1(P_i^t)$. Thus, we know that the only reversal between P_i^{t-1} and P_i^t is of a and b, and hence by Observation 2, $\varphi_b(P_i^t, P_{-i}) > \varphi_b(P_i^{t-1}, P_{-i})$, $\varphi_a(P_i^t, P_{-i}) < \varphi_a(P_i^{t-1}, P_{-i})$ and $\varphi_c(P_i^t, P_{-i}) = \varphi_c(P_i^{t-1}, P_{-i})$ for all $c \notin \{a, b\}$. Consequently, if aP_ib, the conclusion follows from Observation 1. For the remainder of the argument, we assume bP_ia.

Let b be the q^{th}-ranked alternative in P_i, i.e. $b = r_q(P_i)$ where $1 \leq q < |A|$. Pick an arbitrary integer K between 1 and $|A|$. We will show $\sum_{s=1}^{K} \varphi_{r_s(P_i)}(P_i, P_{-i}) \geq \sum_{s=1}^{K} \varphi_{r_s(P_i)}(P_i^t, P_{-i})$ thereby establishing $\varphi(P_i, P_{-i}) P_i^{sd} \varphi(P_i^t, P_{-i})$. We consider two cases.

Suppose $1 \leq K < q$. Then the alternatives ranked above the K^{th}-ranked alternative in P_i do not involve either a or b. By virtue of Part (iii) of Observation 2, the total probability on these alternatives is unchanged between $\varphi(P_i^{t-1}, P_{-i})$ and $\varphi(P_i^t, P_{-i})$. In conjunction with the induction hypothesis, we have $\sum_{s=1}^{K} \varphi_{r_s(P_i)}(P_i, P_{-i}) \geq \sum_{s=1}^{K} \varphi_{r_s(P_i)}(P_i^t, P_{-i})$ as required.

Suppose $q \leq K \leq |A|$. Pick an arbitrary $c \in A$ such that bP_ic. Since $b = r_1(P_i^t)$, we must have bP_i^tc. Property P implies the existence of a path $\bar{\pi} \in \Pi(P_i, P_i^t)$ such that $b\bar{P}_i^r c$ for all $\bar{P}_i^r$ along the path $\bar{\pi}$. Hence, $r_1(\bar{P}_i^r) \neq c$ for all $\bar{P}_i^r$ along $\bar{\pi}$. Applying Lemma 2, we can conclude $\varphi_c(P_i, P_{-i}) = \varphi_c(P_i^t, P_{-i})$. Consequently the total probability of alternatives ranked strictly below the K^{th}-ranked alternative in P_i is the same in $\varphi(P_i, P_{-i})$ and $\varphi(P_i^t, P_{-i})$. Equivalently, the total probability of alternatives ranked above the K^{th}-ranked alternative in P_i is the same in $\varphi(P_i, P_{-i})$ and $\varphi(P_i^t, P_{-i})$, i.e. $\sum_{s=1}^{K} \varphi_{r_s(P_i)}(P_i, P_{-i}) \geq \sum_{s=1}^{K} \varphi_{r_s(P_i)}(P_i^t, P_{-i})$. This completes the proof. ■

Theorem 1 leads immediately to the following corollary.

COROLLARY 1 *If a domain satisfies Property P, it satisfies local-global equivalence for unanimous DSCFs.*

The arguments in the proof of Theorem 1 can be used to show that any locally strategy-proof and unanimous RSCF defined on a domain satisfying Property P also satisfies the important property of *tops-onlyness*.[10]

DEFINITION 7 *A RSCF* $\varphi : \mathcal{D}^n \to \Delta(A)$ *satisfies the tops-only property if for all* $P, P' \in \mathcal{D}^n$, *we have* $[r_1(P_i) = r_1(P_i')$ *for all* $i \in N] \Rightarrow [\varphi(P) = \varphi(P')]$.

Suppose a RSCF satisfies the tops-only property. Then its value at any profile depends only on the peaks of the agent preferences in the profile.

[10]See Chatterji and Sen (2011) and Chatterji and Zeng (2018) for a discussion of this property.

Corollary 2 *If the domain $\mathcal{D}$ satisfies Property P, every unanimous and locally strategy-proof RSCF $\varphi : \mathcal{D}^n \rightarrow \Delta(A)$ satisfies the tops-only property.*

Proof: Fix a unanimous and locally strategy-proof RSCF $\varphi : \mathcal{D}^n \rightarrow \Delta(A)$. To verify the tops-only property, it suffices to show that for all $i \in N$, $P_i, P_i' \in \mathcal{D}$ and $P_{-i} \in \mathcal{D}^{n-1}$, $[r_1(P_i) = r_1(P_i')] \Rightarrow [\varphi(P_i, P_{-i}) = \varphi(P_i', P_{-i})]$.

Pick $i \in N$, $P_i, P_i' \in \mathcal{D}$ and $P_{-i} \in \mathcal{D}^{n-1}$ such that $r_1(P_i) = r_1(P_i') \equiv x$. If $P_i \sim P_i'$, Lemma 1 immediately implies $\varphi(P_i, P_{-i}) = \varphi(P_i', P_{-i})$. Suppose it is not the case that $P_i \sim P_i'$. To show $\varphi(P_i, P_{-i}) = \varphi(P_i', P_{-i})$, it suffices to show $\varphi_a(P_i, P_{-i}) = \varphi_a(P_i', P_{-i})$ for all $a \in A \backslash \{x\}$. Pick $a \in A\backslash\{x\}$. Since $r_1(P_i) = r_1(P_i') = x \neq a$, Property P implies the existence of a path $\pi \in \Pi(P_i, P_i')$ such that $xP_i^r a$ for all P_i^r along the path π. Thus, $r_1(P_i^r) \neq a$ for all P_i^r along the path π. Then, Lemma 2 implies $\varphi_a(P_i, P_{-i}) = \varphi_a(P_i', P_{-i})$. This also implies $\varphi_x(P_i, P_{-i}) = \varphi_x(P_i', P_{-i})$ so that $\varphi(P_i, P_{-i}) = \varphi(P_i', P_{-i})$, as required. ■

Corollary 2 generalizes Theorem 1 of Chatterji and Zeng (2018) on domains satisfying Property P. Their strategy-proofness is weakened to local strategy-proofness, their Interior property becomes redundant and the requirement of their Exterior property is met by Property P. For instance, the domain in Example 1 violates the Interior property but satisfies Property P.

4.1 Group Strategy-proofness

Our goal in this subsection is to show that when turning to the deterministic setting, any locally strategy-proof, unanimous DSCF defined on a domain satisfying Property P also satisfies the stronger property of group strategy-proofness, i.e. no coalition of agents can strictly improve by a joint misrepresentation of their preferences.[11] We denote a coalition by $S \subseteq N$ where S is non-empty. A preference profile for the coalition S is denoted by P_S and a preference profile $P \in \mathcal{D}^n$ is written as (P_S, P_{-S}).

Definition 8 *A DSCF $f : \mathcal{D}^n \rightarrow A$ is group manipulable by a coalition $S \subseteq N$ at profile $P = (P_S, P_{-S})$ if there exists $P_S' \in \mathcal{D}^{|S|}$ such that $f(P_S', P_{-S}) P_i f(P_S, P_{-S})$ for all $i \in S$. The DSCF is group strategy-proof if it is not group manipulable by any coalition at any profile.*

[11]In the random setting, the notion of group strategy-proofness is too demanding. For instance, Corollary 1 of Morimoto (2020) implies that "most" unanimous and strategy-proof RSCFs defined on the domain of single-peaked preferences, which of course satisfies Property P, are group manipulable.

Our main result in this section is the following.

THEOREM **2** *If the domain $\mathcal{D}$ satisfies Property P, every unanimous and locally strategy-proof DSCF is group strategy-proof.*

Proof: Let $\overline{A} = \{a \in A : r_1(P) = a \text{ for some } P \in \mathcal{D}\}$ be the set of alternatives that are first-ranked for some preferences in $\mathcal{D}$. Recall that $\mathcal{D}$ is assumed to contain at least two preferences with distinct peaks. Hence, $|\overline{A}| \geq 2$. Fix a unanimous and locally strategy-proof DSCF $f : \mathcal{D}^n \to A$. The range of f is defined as $R(f) = \{a \in A : f(P) = a \text{ for some } P \in \mathcal{D}^n\}$. Unanimity implies $\overline{A} \subseteq R(f)$. Lemmas 1, 2 and Corollary 1 hold for f, i.e. f is strategy-proof.

LEMMA **3** $R(f) = \overline{A}$.

Proof: Suppose not, i.e. there exists $P = (P_1, P_2, \ldots, P_n) \in \mathcal{D}^n$ such that $f(P) = a \notin \overline{A}$. Let $r_1(P_1) = x$. Thus $x \in \overline{A}$. Let $P' = (P'_1, P'_2, \ldots, P'_n) \in \mathcal{D}^n$ be a preference profile such that $P'_i = P_1$ for all $i \in N$. For each $i \in \{2, \ldots, n\}$, we pick an arbitrary path $\pi_i \in \Pi(P_i, P'_i)$.[12] Since $a \notin \overline{A}$, there does not exist any preference P_i^r in the path π_i with $r_1(P_i^r) = a$. We can move from P to P' by changing P_i to P'_i for each i ranging from $i = 2$ to $i = n$. According to paths $\pi_2, \ldots, \pi_n$, by repeatedly applying Lemma 2, we have $f(P') = f(P) = a \neq x$ which contradicts unanimity. Therefore $R(f) = \overline{A}$. ■

In order to prove the theorem, we will prove the following equivalent reformulation of group strategy-proofness: for all $S \subseteq N$, $P_S, P'_S \in \mathcal{D}^{|S|}$ and $P_{-S} \in \mathcal{D}^{|N \setminus S|}$, either $f(P_S, P_{-S}) = f(P'_S, P_{-S})$ or $f(P_S, P_{-S}) P_i f(P'_S, P_{-S})$ for some $i \in S$.

We will prove this by induction on the cardinality of S. The case where $|S| = 1$ reduces to strategy-proofness which is implied by Corollary 1. Assume that the statement above holds for all $S \subseteq N$ such that $|S| \leq t - 1 < n$. We will show that the statement holds for all $S \subseteq N$ where $|S| = t$.

Suppose not, i.e. there exists $S \subseteq N$ (with $|S| = t$) such that $f(P'_S, P_{-S}) = b$, $f(P_S, P_{-S}) = a$ and $b P_i a$ for all $i \in S$. Since $b \in R(f)$, Lemma 3 implies that there exists $P_i^* \in \mathcal{D}$ such that $r_1(P_i^*) = b$. Furthermore, since $f(P'_S, P_{-S}) = b$, strategy-proofness implies $f(P_S^*, P_{-S}) = b$ where every voter of S has the preference P_i^*.

Since $f(P_S, P_{-S}) = a \neq b = f(P_S^*, P_{-S})$, we have a voter $j \in S$ such that $P_j \neq P_i^*$. By Property P, we have a path $\pi = (P_j^1, \ldots, P_j^v) \in \Pi(P_j, P_i^*)$ with no $\{a, b\}$-restoration. Since

[12] If $P_i = P'_i$, π_i is the null path that begins and terminates at P_i.

bP_ja and bP_i^*a, no $\{a,b\}$-restoration on π implies bP_j^ka for all $k = 1, \ldots, v$. Hence, $r_1(P_j^k) \neq a$ for all $k = 1, \ldots, v$. Since $f(P_j, P_{S\setminus\{j\}}, P_{-S}) = a$, Lemma 2 implies $f(P_i^*, P_{S\setminus\{j\}}, P_{-S}) = a$.[13] Since $f(P_i^*, P_{S\setminus\{j\}}, P_{-S}) = a$ and $f(P_i^*, P^*_{S\setminus\{j\}}, P_{-S}) = b$, coalition $S\setminus\{j\}$ can group manipulate at profile $(P_i^*, P_{S\setminus\{j\}}, P_{-S})$, which contradicts the induction hypothesis. This completes the proof. ■

There are some papers that investigate preference domains on which equivalence of strategy-proofness and group strategy-proofness holds. Barberà et al. (2010) consider a more general setting than ours in the following respects: (i) the alternative set is either finite or infinite, (ii) preferences can admit indifference, (iii) preference domains can vary across different voters, and (iv) unanimity is not exogenously imposed on DSCFs. On the other hand, our result has a weaker premise — local strategy-proofness rather than strategy-proofness. In addition our Property P is far simpler (especially in the computational sense) than their *sequential inclusion condition*.[14] The latter is a condition imposed on preference profiles while Property P is a condition imposed only on preferences in a domain. Our result is not implied by theirs - for example, the domain of single-peaked preferences on a tree introduced by Demange (1982) is covered by our condition but not by theirs.

Property P is also independent of the sufficient condition identified in Le Breton and Zaporozhets (2009) for the equivalence of strategy-proofness and group strategy-proofness. For instance, consider a domain $\mathcal{D}$ consisting of the three preferences $P_i^1 = (a\,b\,c\,d)$, $P_i^2 = (a\,b\,d\,c)$ and $P_i^3 = (b\,a\,d\,c)$.[15] This domain satisfies Property P but violates the richness condition of Le Breton and Zaporozhets (2009) — though bP_i^1c and cP_i^1d, there exists no preference $P_i \in \mathcal{D}$ such that $r_1(P_i) = b$ and cP_id.

5 Necessity

We have already shown that Property P guarantees uLGE and ensures that in the deterministic setting, local strategy-proofness implies group strategy-proofness. However, it is not a necessary condition for uLGE as Example 2 shows.

[13] Here agent j has the preference P_i^* in the profile $(P_i^*, P_{S\setminus\{j\}}, P_{-S})$.

[14] According to Section 4.1 of Kumar et al. (2020), verifying whether Property L, which as mentioned is significantly stronger than Property P, is satisfied is not computationally hard.

[15] For notational convenience, we specify preferences here horizontally. For instance, $P_i^1 = (a\,b\,c\,d)$ represents that a is top-ranked, b is second-ranked, c is third-ranked, and d is bottom-ranked.

Example 2 Let $A = \{a, b, c, d, x, y\}$. The domain $\mathcal{D}$ is specified in Table 2 and Figure 2 illustrates the path induced by the adjacent preferences in $\mathcal{D}$. Note that there is a single path between P_i^1 and P_i^6 which has a $\{b, c\}$-restoration. It follows that $\mathcal{D}$ violates Property P.

P_i^1	P_i^2	P_i^3	P_i^4	P_i^5	P_i^6
a	a	a	a	a	b
b	c	c	c	b	a
c	b	b	b	c	c
d	d	d	y	y	y
x	x	y	d	d	d
y	y	x	x	x	x

Table 2: The Domain $\mathcal{D}$

$$P_i^1 \overset{\{b,c\}}{\text{———}} P_i^2 \overset{\{x,y\}}{\text{———}} P_i^3 \overset{\{d,y\}}{\text{———}} P_i^4 \overset{\{c,b\}}{\text{———}} P_i^5 \overset{\{a,b\}}{\text{———}} P_i^6$$

Figure 2: The path induced by the adjacent preferences in $\mathcal{D}$

Let $\varphi : \mathcal{D}^n \to \Delta(A)$ be an arbitrary unanimous and locally strategy-proof RSCF. Observe that the first-ranked alternatives in each of the preferences in $\mathcal{D}$ is either a or b. In order for φ to satisfy unanimity, it must be the case that a and b exhaust the whole probability at each preference profile, i.e. $\varphi_a(P) + \varphi_b(P) = 1$ for all $P \in \mathcal{D}^n$.[16] Consequently, at profiles (P_i, P_{-i}) and (P_i', P_{-i}) such that $P_i \sim P_i'$, zP_iz', $z'P_i'z$ and $\{z, z'\} \neq \{a, b\}$, we have $\varphi(P_i, P_{-i}) = \varphi(P_i', P_{-i})$. Therefore, the $\{b, c\}$-restoration alluded to earlier, is irrelevant. Finally, since the path of Figure 2 has no $\{a, b\}$-restoration, it is easy to show that $\mathcal{D}$ satisfies uLGE following the proof of Theorem 1. □

Hong and Kim (2020) restrict attention to DSCFs, focus on ordinal Bayesian incentive compatibility, and establish uLGE for domains satisfying a property called *Sparsely Connected Domain without Restoration* (or SCD). This property requires the existence of paths without

[16] Let $\mathcal{D}^1 = \{P_i^1, P_i^2, P_i^3, P_i^4, P_i^5\}$ and $\mathcal{D}^2 = \{P_i^6\}$. Given a profile $P \in \mathcal{D}^n$, if $P_i \in \mathcal{D}^1$ for all $i \in N$, unanimity implies $\varphi_a(P) = 1$. Symmetrically, $\varphi_b(P) = 1$ if $P_i \in \mathcal{D}^2$ for all $i \in N$. Suppose $\varphi_z(P) > 0$ for some $z \in A\backslash\{a, b\}$ and some $P \in \mathcal{D}^n$. It must be the case that $P_i \in \mathcal{D}^1$ and $P_j \in \mathcal{D}^2$ for some $i, j \in N$. Assume for notational convenience that $P_i \in \mathcal{D}^1$ for all $i = 1, \ldots, s$ and $P_j = P_i^6$ for all $j = s + 1, \ldots, n$, where $1 \leq s < n$. Let $P_{s+1}' = \cdots = P_n' = P_i^5$ and $P^\ell = (P_1, \ldots, P_s, P_{s+1}', \ldots, P_\ell', P_{\ell+1}, \ldots, P_n)$ for all $\ell = s + 1, \ldots, n$. Thus, $P_j' \sim P_j$ for all $j = s + 1, \ldots, n$, and unanimity implies $\varphi_a(P^n) = 1$ and hence $\varphi_z(P^n) = 0$. Consequently, Observation 2 implies $0 = \varphi_z(P^n) = \cdots = \varphi_z(P^{s+1}) = \varphi_z(P) > 0$. Contradiction.

restoration for all pairs of alternatives such that at least one of the two alternatives is first-ranked in some preference in the domain. This condition is slightly weaker than Property P since the no-restoration requirement is imposed only on a subset of all pairs of alternatives. However, SCD is not necessary for uLGE either. For instance, the domain in Example 2 violates SCD because the path of Figure 2 has a $\{b,c\}$-restoration and b is first-ranked in P_i^6.

A characterization of domains that satisfy uLGE remains an open problem. However, we are able to show that uLGE implies connectedness of a domain.

Proposition 1 *If a domain satisfies uLGE, it is connected.*

Proof: Pick a domain $\mathcal{D}$ that satisfies uLGE. Suppose that domain $\mathcal{D}$ is not connected. We can then partition $\mathcal{D}$ into two non-empty subsets $\mathcal{D}^1$ and $\mathcal{D}^2$ such that there does not exist any $P_i \in \mathcal{D}^1$ and $P_i' \in \mathcal{D}^2$ with $P_i \sim P_i'$.

There are several cases to consider. In each one, we find a set of agents and construct a unanimous, locally strategy-proof and manipulable DSCF. We begin with an observation that we will use frequently.

Observation 3 We consider a particular class of DSCFs in this setting. We say that a DSCF f is *local* if for all $i \in N$, $P_{-i} \in \mathcal{D}^{n-1}$, $j \in \{1,2\}$ and $P_i, P_i' \in \mathcal{D}^j$,

$$[f(P_i,P_{-i}) \neq f(P_i',P_{-i})] \Rightarrow [f(P_i,P_{-i}) = r_1(P_i) \text{ and } f(P_i',P_{-i}) = r_1(P_i')].$$

Suppose that agent i's true preference is $P_i \in \mathcal{D}^j$ for some $j \in \{1,2\}$. A local misrepresentation of P_i is some preference P_i' that also belongs to $\mathcal{D}^j$. Thus local DSCFs are locally strategy-proof.

Case 1: There exist $\bar{P}_i \in \mathcal{D}^1$ and $\hat{P}_i \in \mathcal{D}^2$ such that $r_1(\bar{P}_i) = r_1(\hat{P}_i)$.

Let $N = \{1,2\}$. Consider the following DSCF:

$$f(P_1,P_2) = \begin{cases} r_1(P_1) & \text{if } P_1,P_2 \in \mathcal{D}^1 \text{ or } P_1,P_2 \in \mathcal{D}^2, \text{ and} \\ r_1(P_2) & \text{otherwise.} \end{cases}$$

The outcome at each preference profile is the first-ranked alternative of some voter's preference; it is evident that f is unanimous. It is easy to verify that f is local.[17] Then,

[17]For agent 1, pick $P_1, P_1' \in \mathcal{D}^j$ for some $j \in \{1,2\}$ and $P_2 \in \mathcal{D}$. If $f(P_1,P_2) \neq f(P_1',P_2)$, we can deduce that $P_2 \in \mathcal{D}^j$. Hence $f(P_1,P_2) = r_1(P_1)$ and $f(P_1',P_2) = r_1(P_1')$. For agent 2, fix $P_2, P_2' \in \mathcal{D}^j$ for some $j \in \{1,2\}$ and $P_1 \in \mathcal{D}$. If $f(P_1,P_2) \neq f(P_1,P_2')$, we immediately deduce that $P_1 \notin \mathcal{D}^j$. Hence $f(P_1,P_2) = r_1(P_2)$ and $f(P_1,P_2') = r_1(P_2')$. Therefore, f is local.

Observation 3 implies local strategy-proofness. However, f is not strategy-proof. Suppose $r_1(\bar{P}_i) = r_1(\hat{P}_i) = x$. Recall that $\mathcal{D}$ is assumed to contain at least two preferences with distinct peaks. Therefore, there exists $P_2 \in \mathcal{D}$ such that $r_1(P_2) = y \neq x$. Suppose $P_2 \in \mathcal{D}^2$. Then $f(\bar{P}_i, P_2) = y$ and $f(\hat{P}_i, P_2) = x$.[18] Agent 1 will then manipulate at $(\bar{P}_i, P_2)$ via $\hat{P}_i$. If $P_2 \in \mathcal{D}^1$, we have $f(\bar{P}_i, P_2) = x$ and $f(\hat{P}_i, P_2) = y$. Then, agent 1 will manipulate at $(\hat{P}_i, P_2)$ via $\bar{P}_i$. This contradicts the hypothesis that $\mathcal{D}$ satisfies uLGE.

Case 1 implies that all preferences with the same first-ranked alternative must belong to the same subset of $\mathcal{D}$, i.e. $[P'_i \in \mathcal{D}^j$ and $r_1(P''_i) = r_1(P'_i)] \Rightarrow [P''_i \in \mathcal{D}^j]$, for $j = 1, 2$. Let $\tau(\mathcal{D}^j) = \{a \in A : r_1(P_i) = a$ for some $P_i \in \mathcal{D}^j\}$, for $j = 1, 2$. We consider two cases, labelled Case 2 and 3. In each case, we show the existence of a unanimous, locally strategy-proof and manipulable DSCF.

Case 2: $|\tau(\mathcal{D}^j)| > 1$ for some $j \in \{1, 2\}$.

Assume w.l.o.g. that $|\tau(\mathcal{D}^2)| > 1$. Let $x, y \in \tau(\mathcal{D}^2)$ and $P^*_i \in \mathcal{D}^1$. Assume w.l.o.g. that $xP^*_i y$. Let $N = \{1, 2\}$. Consider the following DSCF:

$$f(P_1, P_2) = \begin{cases} r_1(P_1) & \text{if } P_1, P_2 \in \mathcal{D}^1 \text{ or } P_1, P_2 \in \mathcal{D}^2, \text{ and} \\ y & \text{otherwise.} \end{cases}$$

Let (P_1, P_2) be a profile such that $r_1(P_1) = r_1(P_2)$. By virtue of our assumption, it must be the case that $P_1, P_2 \in \mathcal{D}^j$, for some $j \in \{1, 2\}$. Since f picks an agent's first-ranked alternative in such a profile, it is clear that f satisfies unanimity. Again f is local.[19] So Observation 3 implies that f is locally strategy-proof. Finally, we show that f is not strategy-proof. Since $x \in \tau(\mathcal{D}^2)$, there exists $P_1 \in \mathcal{D}^2$ with $r_1(P_1) = x$. By construction, $f(P_1, P^*_i) = y$ and $f(P_1, P_1) = x$. Since $xP^*_i y$, agent 2 manipulates at (P_1, P^*_i) via P_1. Therefore Case 2 cannot occur.

Case 3: $|\tau(\mathcal{D}^1)| = |\tau(\mathcal{D}^2)| = 1$.

Let $\tau(\mathcal{D}^1) = \{x\}$ and $\tau(\mathcal{D}^2) = \{y\}$. Recall that $|A| \geq 3$. Accordingly, we consider two subcases: (A) there exists $P^*_i \in \mathcal{D}$ such that $r_{|A|}(P^*_i) = z \notin \{x, y\}$, and (B) $r_{|A|}(P_i) \in \{x, y\}$ for all $P_i \in \mathcal{D}$.

[18]Here $(\bar{P}_i, P_2)$ is the profile where agent 1's preference is $\bar{P}_i$ and agent 2's preference is P_2. Similarly $(\hat{P}_i, P_2)$ is the profile where agent 1's preference is $\hat{P}_i$ and agent 2's preference is P_2.

[19] Arguing as we did in Footnote 17, by picking $P_1, P'_1 \in \mathcal{D}^j$ for some $j \in \{1, 2\}$ and $P_2 \in \mathcal{D}$, we can infer $[f(P_1, P_2) \neq f(P'_1, P_2)] \Rightarrow [f(P_1, P_2) = r_1(P_1)$ and $f(P'_1, P_2) = r_1(P'_1)]$. Next, fixing $P_2, P'_2 \in \mathcal{D}^j$ for some $j \in \{1, 2\}$ and $P_1 \in \mathcal{D}$, we always have $f(P_1, P_2) = f(P_1, P'_2)$ by the construction of f. Therefore, f is local.

Case 3A: Assume w.l.o.g. that $r_1(P_i^*) = x$, i.e. $P_i^* \in \mathcal{D}^1$. By assumption, yP_i^*z. Let $N = \{1, 2\}$ and consider the following DSCF:

$$f(P_1, P_2) = \begin{cases} x & \text{if } P_1, P_2 \in \mathcal{D}^1, \\ y & \text{if } P_1, P_2 \in \mathcal{D}^2, \text{ and} \\ z & \text{otherwise.} \end{cases}$$

It is easy to verify that f satisfies unanimity. Local strategy-proofness follows again from Observation 3 as f is local.[20] Again f is not strategy-proof. Pick $P_2 \in \mathcal{D}^2$. By construction $f(P_i^*, P_2) = z$ and $f(P_2, P_2) = y$. Since yP_i^*z, agent 1 manipulates at (P_i^*, P_2) via P_2.

Case 3B: Since $|A| \geq 3$, there must exist $z \in A\backslash\{x, y\}$ and $\hat{P}_i \in \mathcal{D}$ such that $z\hat{P}_iy$ or $z\hat{P}_ix$ holds. We assume w.l.o.g. that $z\hat{P}_iy$. Thus $\hat{P}_i \in \mathcal{D}^1$. Let $N = \{1, 2, 3\}$ and consider the following DSCF.

$$f(P_1, P_2, P_3) = \begin{cases} x & \text{if } P_1, P_2, P_3 \in \mathcal{D}^1, \\ y & \text{if } P_1, P_2, P_3 \in \mathcal{D}^2, \\ y & \text{if } P_i \in \mathcal{D}^2 \text{ for some } i \in \{1, 2, 3\} \text{ and } P_j \in \mathcal{D}^1 \text{ for all } j \neq i, \text{ and} \\ z & \text{if } P_i \in \mathcal{D}^1 \text{ for some } i \in \{1, 2, 3\} \text{ and } P_j \in \mathcal{D}^2 \text{ for all } j \neq i. \end{cases}$$

In order to show unanimity, we need to only consider profiles where all agents have preferences belonging to the same $\mathcal{D}^j$. In each of these cases, f picks the commonly first-ranked alternative. Also f is local,[21] and we can deduce that f is locally strategy-proof from Observation 3. Finally we show that f is not strategy-proof. Consider the profile $(\hat{P}_i, \hat{P}_i, P_3)$ where voters 1 and 2 report the preference $\hat{P}_i$, and voter 3 reports a preference $P_3 \in \mathcal{D}^2$. By construction, $f(\hat{P}_i, \hat{P}_i, P_3) = y$. Consider another profile $(\hat{P}_i, P_3, P_3)$ where voter 1 reports the preference $\hat{P}_i$, and voters 2 and 3 reports the preference P_3. By construction, $f(\hat{P}_i, P_3, P_3) = z$. Consequently agent 2 will manipulate at $(\hat{P}_i, \hat{P}_i, P_3)$ via P_3 since $z\hat{P}_iy$.

This concludes the proof of Proposition 1. ■

[20] Fixing $P_1, P_1' \in \mathcal{D}^j$ for some $j \in \{1, 2\}$ and $P_2 \in \mathcal{D}$, we always have $f(P_1, P_2) = f(P_1', P_2)$ by the construction of f. Symmetrically, fixing $P_2, P_2' \in \mathcal{D}^j$ for some $j \in \{1, 2\}$ and $P_1 \in \mathcal{D}$, we also have $f(P_1, P_2) = f(P_1, P_2')$ by the construction of f. Therefore, f is local vacuously.

[21] Fixing arbitrary $i \in N$, $P_i, P_i' \in \mathcal{D}^j$ for some $j \in \{1, 2\}$ and $P_{-i} \in \mathcal{D}^{n-1}$, it is easy to show that $f(P_i, P_{-i}) = f(P_i', P_{-i})$ by the construction of f. Therefore, f is local vacuously.

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